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by

Philippe Briet, Horia D. Cornean and Delphine Louis

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Diamagnetic expansions for perfect quantum gases

April 7, 2006

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Abstract: In this work we study the diamagnetic properties of a perfect quantum gas in the presence of a constant magnetic field of intensity B . We investigate the Gibbs semigroup associated to the one particle operator at finite volume, and study its Taylor series with respect to the field parameter $\omega := eB/c$ in different topologies. This allows us to prove the existence of the thermodynamic limit for the pressure and for all its derivatives with respect to ω (the so-called generalized susceptibilities).

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Keywords: semigroup, magnetic field, thermodynamic limit.

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1 Introduction and results.

This paper is motivated by the study of the diamagnetic properties of a perfect quantum gas interacting with a constant magnetic field, $\mathbf{B} := B\mathbf{e}_3, B > 0$, $\mathbf{e}_3 := (0, 0, 1)$. The system obeys either the Bose or the Fermi statistics. Since

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we are only studying orbital diamagnetic effects, we consider a gas of spinless and charged particles.

We are mainly interested in the bulk response i.e. the thermodynamic limit of the pressure and its derivatives w.r.t. cyclotron frequency $\omega := eB/c$. As in [BCL] we use the term *generalized susceptibilities* to designate such quantities.

From a rigorous point of view, this question was already studied by several authors. There are results concerning the existence of the large volume limit of the pressure for both Fermi and Bose gases [AC, ABN2], the magnetization for a Bose gas [C, MMP] and the magnetic susceptibility for a Fermi gas [ABN2]. In [BCL], extensions of these results to the case of generalized susceptibilities were announced.

This paper is the first in a series of two devoted to the rigorous proof of the results announced in [BCL]. Here we consider the regime in which the inverse temperature $\beta := 1/(kT)$ is positive and finite and the fugacity $z = e^{\beta\mu}$ belongs to the unit complex disk. Such conditions were also used in [AC, ABN2, MMP]. But here we also allow any positive value of the cyclotron frequency $\omega := e/cB$. In a forthcoming paper we will extend these results to some larger z -complex domains (in fact to \mathbf{D}_ϵ defined below). One can also find different aspects of this problem in [MMP, CoRo, HeSj].

The main part of this work is concerned with a new approach to the magnetic perturbation theory for a semigroup generated by a magnetic Schrödinger operator. It extends the results given in [ABN1, C] and heavily relies on the use of the magnetic phase factor. This allows us to have a good control on the magnetic perturbation w.r.t. the size of the volume [BC, CN] in which the gas is confined.

Let us now describe our results. Let Λ be an open, bounded and connected subset of \mathbb{R}^3 containing the origin of \mathbb{R}^3 and with smooth boundary $\partial\Lambda$. We also suppose that Λ has a diameter $\text{diam}(\Lambda) \leq 1$. Set

$$\Lambda_L := \{\mathbf{x} \in \mathbb{R}^3, \mathbf{x}/L \in \Lambda\}; L > 1. \quad (1.1)$$

Here we use the transverse gauge i.e. the magnetic potential is defined as $\mathbf{Ba} := (B/2)\mathbf{e}_3 \wedge \mathbf{x}$. The one particle Hamiltonian

$$H_L(\omega) = \frac{1}{2}(-i\nabla - \omega\mathbf{a})^2, \quad (1.2)$$

is first defined in the form sense on $H_0^1(\Lambda_L)$, and then one considers its Friedrichs extension. Thus we work with Dirichlet boundary conditions (DBC).

It is well-known that $H_L(\omega)$, $\omega \in \mathbb{R}$, generates a Gibbs semigroup

$$\{W_L(\beta, \omega) = e^{-\beta H_L(\omega)} : \beta \geq 0\} \quad (1.3)$$

i.e. for all $\beta > 0$, $W_L(\beta, \omega) \in B_1(L^2(\mathbb{R}^3))$, the set of trace class operators on \mathcal{H}_L [AC, Z].

Then for $\beta > 0$, $\omega \in \mathbb{R}$, the grand canonical pressure of a quantum gas at finite volume is defined as ([Hu, AC, ABN2])

$$P_L(\beta, \omega, z, \epsilon) = \frac{\epsilon}{\beta|\Lambda_L|} \cdot \text{Tr} \{ \ln(1 + \epsilon z W_L(\beta, \omega)) \}, \quad (1.4)$$

where $\epsilon = -1$ ($\epsilon = 1$) for Bose (Fermi) statistics. Since $\omega/2 = \inf \sigma(H_\infty(\omega))$, then the pressure is an analytic function w.r.t. z on the complex domain \mathbf{D}_ϵ

with

$$\mathbf{D}_{+1} := \mathbb{C} \setminus (-\infty, -e^{\beta\omega/2}], \quad \mathbf{D}_{-1} := \mathbb{C} \setminus [e^{\beta\omega/2}, \infty).$$

Let $n \geq 1$ and define the susceptibility of order n at finite volume by:

$$\chi_L^{(n)}(\beta, \omega, z, \epsilon) := \frac{\partial^n P_L}{\partial \omega^n}(\beta, \omega, z, \epsilon); \quad (1.5)$$

If $n = 0$ we set $\chi_L^{(0)}(\beta, \omega, z, \epsilon) := P_L(\beta, \omega, z, \epsilon)$.

Our first result describes the properties of the above defined quantities at finite volume, and it is given by the following theorem:

Theorem 1.1. *Let $\beta > 0$. Then the map $\mathbb{R} \ni \omega \rightarrow W_L(\beta, \omega) \in B_1(L^2(\mathbb{R}^3))$ is real analytic. For each open set \mathcal{K} whose closure is compact and $\overline{\mathcal{K}} \subset \mathbf{D}_\epsilon$, $\epsilon = -1, +1$, there exists an open neighborhood \mathcal{N} of the real axis such that the pressure at finite volume $P_L(\beta, \omega, z, \epsilon)$ is analytic w.r.t. (ω, z) on $\mathcal{N} \times \mathcal{K}$. Let $\omega \in \mathcal{N} \cap \mathbb{R}$, and $|z| < 1$. Then for $n \geq 0$ we have (see (1.3)):*

$$\chi_L^{(n)}(\beta, \omega, z, \epsilon) = \frac{\epsilon}{\beta|\Lambda_L|} \sum_{k \geq 1} \frac{(-\epsilon z)^k}{k} \operatorname{Tr} \left\{ \frac{\partial^n W_L(k\beta, \omega)}{\partial \omega^n} \right\}. \quad (1.6)$$

We now discuss the limit $L = \infty$. First we define the candidates, $\chi_\infty^{(n)}$, for these limits. Recall that the one particle operator $H_\infty(\omega) = \frac{1}{2}(-i\nabla - \omega \mathbf{a})^2$ on $L^2(\mathbb{R}^3)$ is positive and essentially selfadjoint on $C_0^\infty(\mathbb{R}^3)$. Denote by $W_\infty(\beta, \omega), \beta \geq 0$ the semigroup generated by $H_\infty(\omega)$. Then $W_\infty(\beta, \omega)$ has an explicit integral kernel satisfying (see Section 3):

$$G_\infty(\mathbf{x}, \mathbf{x}; \beta, \omega) = \frac{1}{(2\pi\beta)^{3/2}} \frac{\omega\beta/2}{\sinh(\omega\beta/2)}, \quad \forall \mathbf{x} \in \mathbb{R}^3. \quad (1.7)$$

Note that the right hand side is independent of \mathbf{x} . Let $\beta > 0, \omega \geq 0$ and $|z| < 1$. In view of (1.6), define:

$$P_\infty(\beta, \omega, z, \epsilon) := \frac{\epsilon}{\beta} \sum_{k \geq 1} \frac{(-\epsilon z)^k}{k} G_\infty(\mathbf{0}, \mathbf{0}; k\beta, \omega), \quad (1.8)$$

which is well defined because of the estimate $\sinh(t) \geq t$ if $t \geq 0$. Then by the results of [AC, ABN2], we know that

$$\lim_{L \rightarrow \infty} P_L(\beta, \omega, z, \epsilon) = P_\infty(\beta, \omega, z, \epsilon).$$

It is quite natural to choose $\chi_\infty^{(n)} := \frac{\partial^n P_\infty}{\partial \omega^n}$ provided that this last quantity exists. Note that it is not very easy to see this just from (1.8) and (1.7).

We then prove the following

Theorem 1.2. *Let $\beta > 0, \omega \geq 0$ and $|z| < 1$. Fix $n \geq 1$ and define*

$$\chi_\infty^{(n)}(\beta, \omega, z, \epsilon) := \frac{\partial^n P_\infty}{\partial \omega^n}(\beta, \omega, z, \epsilon). \quad (1.9)$$

Then we have the equality :

$$\chi_\infty^{(n)}(\beta, \omega, z, \epsilon) = \frac{\epsilon}{\beta} \sum_{k \geq 0} \frac{(-\epsilon z)^k}{k} \frac{\partial^n G_\infty}{\partial \omega^n}(\mathbf{0}, \mathbf{0}; k\beta, \omega). \quad (1.10)$$

Moreover,

$$\lim_{L \rightarrow \infty} \chi_L^{(n)}(\beta, \omega, z, \epsilon) = \chi_\infty^{(n)}(\beta, \omega, z, \epsilon) \quad (1.11)$$

uniformly on $[\beta_0, \beta_1] \times [\omega_0, \omega_1]$, $0 < \beta_0 < \beta_1 < \infty$ and $0 < \omega_0 < \omega_1 < \infty$.

1.1 Relation with the de Haas-van Alphen (dHvA) effect

Our results can be easily extended to the case of more general Bloch electrons, that is when one has a background smooth and periodic electric potential V . More precisely, $V \in C^\infty(\mathbb{R}^3)$, $V \geq 0$, and if Γ is a periodic lattice in \mathbb{R}^3 then $V(\cdot) = V(\cdot + \gamma)$ for all $\gamma \in \Gamma$. Denote by Ω the elementary cell of Γ . In this case, the grandcanonical pressure at the thermodynamic limit will be given by (we work with fermions thus $\epsilon = 1$)

$$P_\infty(\beta, \omega, z) = \frac{1}{\beta} \sum_{k \geq 1} \frac{(-z)^k}{k} \frac{1}{|\Omega|} \int_\Omega G_\infty(\mathbf{x}, \mathbf{x}; k\beta, \omega) d\mathbf{x}, \quad (1.12)$$

where $G_\infty(\mathbf{x}, \mathbf{x}'; k\beta, \omega)$ is the smooth integral kernel of the semigroup generated by $\frac{1}{2}(-i\nabla + \omega \mathbf{a})^2 + V$. This formula only holds for $|z| < 1$, but it can be analytically continued to $\mathbb{C} \setminus (-\infty, -1]$, see [AC] or [HeSj].

Now one can start looking at the behavior of $P_\infty(\beta, \omega, z)$ as function of ω , in particular around the point $\omega_0 = 0$. Working in canonical conditions, that is when z is a function of β, ω and the fixed particle density ρ , then one is interesting in the object

$$p_\infty(\beta, \omega, \rho) := P_\infty(\beta, \omega, z(\beta, \omega, \rho)).$$

A thorough analysis of the ω behavior near 0, involving derivatives with respect to ω of the above quantity, has been already given by Helffer and Sjöstrand in [HeSj].

Alternatively, one can start from the finite volume quantities, and define a $z_L(\beta, \omega, \rho)$ as the unique solution of the equation $\rho_L := \beta z \partial_z P_L(\beta, \omega, z_L) = \rho$ and $p_L(\beta, \omega, \rho) := P_\infty(\beta, \omega, z_L(\beta, \omega, \rho))$. Is it still true that at large volumes we have for example that

$$\partial_\omega^n p_L(\beta, \omega, \rho) \sim \partial_\omega^n p_\infty(\beta, \omega, \rho), \quad n \geq 1 ?$$

The main achievement of our paper is that the answer is yes, at least for small densities (which fix $|z| < 1$). In a companion paper we will prove that this is true for all $z \in \mathbb{C} \setminus (-\infty, -1]$.

We end the introduction by giving the plan of this paper. In Section 2 we discuss the analyticity of the Gibbs semigroup with respect to ω in the trace class sense. The trace norm estimates we obtain depend on the size of the domain, due to the linear growth of the magnetic potential. Using magnetic perturbation theory we manage to regularize the trace expansions and to extend these results to the infinite volume case in Sections 3 and 4. Finally we prove the existence of thermodynamic limits in Section 5.

2 Analyticity of Gibbs semigroups

2.1 B_1 -Analyticity

Let Λ_L , $L \geq 1$ be domains of \mathbb{R}^3 as defined in (1.1). In the following we will denote respectively by $\|T\|_1$, $\|T\|_2$ and $\|T\|$, the trace norm in $B_1(L^2(\Lambda_L))$, the Hilbert-Schmidt norm in $B_2(L^2(\Lambda_L))$ and the operator norm in $B_\infty(L^2(\Lambda_L))$ of T .

In this section we study the ω expansion of $W_L(\beta, \omega)$. This question has been already considered [HP, ABN2, Z] in connection with the B_1 analyticity of $W_L(\beta, \omega)$. Combining their result with our analysis below, this gives the following. Define the operators:

$$\hat{R}_{1,L}(\beta, \omega) := \mathbf{a} \cdot (i\nabla_{\mathbf{x}} + \omega \mathbf{a}) W_L(\beta, \omega), \quad (2.1)$$

$$\hat{R}_{2,L}(\beta, \omega) := \frac{1}{2} \mathbf{a}^2 W_L(\beta, \omega). \quad (2.2)$$

Both operators $\hat{R}_{1,L}(\beta, \omega)$ and $\hat{R}_{2,L}(\beta, \omega)$ belong to $B_\infty(\Lambda_L)$ and we have the following estimate on their norm:

Lemma 2.1. *For all $\beta > 0$, $\omega \geq 0$ and $L > 1$, there exists a positive constant C such that:*

$$\|\hat{R}_{1,L}\| \leq \frac{C L}{\sqrt{\beta}} \quad \text{and} \quad \|\hat{R}_{2,L}\| \leq C L^2. \quad (2.3)$$

Proof. Let $\varphi \in L^2(\Lambda_L)$. Since $W_L(\beta, \omega) L^2(\Lambda_L) \subset \text{Dom}(H_L(\omega))$ [K], we have after a standard argument (note that the absolute value of the components of \mathbf{a} are bounded from above by L):

$$\begin{aligned} \|\mathbf{a} \cdot (i\nabla_{\mathbf{x}} + \omega \mathbf{a}) W_L(\beta, \omega) \varphi\|^2 &\leq C L^2 \langle H_L(\omega) W_L(\beta, \omega) \varphi, W_L(\beta, \omega) \varphi \rangle \\ &\leq \frac{C \cdot L^2}{\beta} \|\varphi\|^2, \end{aligned} \quad (2.4)$$

where the last estimate is given by the spectral theorem. The second bound of (2.3) is obvious. \square

Remark 2.2. *Due to the diamagnetic inequality (see (2.30) below), we have for all $\beta > 0$ and $\omega \geq 0$:*

$$\|W_L(\beta, \omega)\|_1 = \text{Tr} (W_L(\beta, \omega)) \leq \frac{L^3}{(2\pi\beta)^{3/2}}. \quad (2.5)$$

Then both operators $\hat{R}_{1,L}, \hat{R}_{2,L}$ are trace class, since we can factorize the operator $\hat{R}_{1,L}(\beta, \omega) = \hat{R}_{1,L}(\beta/2, \omega) W_L(\beta/2, \omega)$.

For $n \geq 1$ define:

$$\mathcal{D}_n(\beta) := \{0 < \tau_n < \tau_{n-1} < \dots < \tau_1 < \beta\} \subset \mathbb{R}^n. \quad (2.6)$$

Let $(i_1, \dots, i_n) \in \{1, 2\}^n$. Lemma 2.1 allows us to define the following family of bounded operators:

$$\begin{aligned} \hat{I}_{n,L}(i_1, \dots, i_n)(\beta, \omega) &:= \int_{\mathcal{D}_n(\beta)} W_L(\beta - \tau_1, \omega) \hat{R}_{i_1,L}(\tau_1 - \tau_2, \omega) \\ &\cdot \hat{R}_{i_2,L}(\tau_2 - \tau_3, \omega) \dots \hat{R}_{i_{n-1},L}(\tau_{n-1} - \tau_n, \omega) \hat{R}_{i_n,L}(\tau_n, \omega) d\tau, \end{aligned} \quad (2.7)$$

where $d\tau$ is the n -dimensional Lebesgue measure. These operators are in fact trace class, and we will estimate their trace norm later. Let $n \geq 1$, $(i_1, \dots, i_n) \in \{1, 2\}^n$ and χ_k^n be the characteristic function,

$$\chi_k^n(i_1, \dots, i_k) := \begin{cases} 1 & \text{if } i_1 + \dots + i_k = n \\ 0 & \text{otherwise.} \end{cases} \quad (2.8)$$

Then we have

Theorem 2.3. *Fix $\beta > 0$. Then the operator-valued function $\mathbb{R} \ni \omega \mapsto W_L(\beta, \omega) \in B_1$ admits an entire extension to the whole complex plane. Fix $\omega_0 \geq 0$. For all $\omega \in \mathbb{C}$ we have*

$$W_L(\beta, \omega) = \sum_{n=0}^{\infty} \frac{(\omega - \omega_0)^n}{n!} \frac{\partial^n W_L}{\partial \omega^n}(\beta, \omega_0), \quad (2.9)$$

$$\frac{1}{n!} \frac{\partial^n W_L}{\partial \omega^n}(\beta, \omega_0) = \sum_{k=1}^n (-1)^k \sum_{i_j \in \{1, 2\}} \chi_k^n(i_1, \dots, i_k) \hat{I}_{k,L}(i_1, \dots, i_k)(\beta, \omega_0). \quad (2.10)$$

Moreover, there exists a positive constant C independent of $n \geq 1$, $\beta > 0$ and L such that:

$$\left\| \frac{1}{n!} \frac{\partial^n W_L}{\partial \omega^n}(\beta, \omega_0) \right\|_1 \leq C^n \frac{(1 + \beta)^n}{\beta^{3/2}} L^{n+3} \frac{1}{[(n-1)/4]!}. \quad (2.11)$$

For all $\omega \in \mathbb{C}$, $\{W_L(\beta, \omega), \beta > 0\}$ is a Gibbs semigroup with its generator given by the closed operator $H_L(\omega)$.

Remark 2.4. *This theorem implies that the trace of the semigroup W_L is an entire function of ω and by (2.9):*

$$\text{Tr}(W_L(\beta, \omega)) = \sum_{n=0}^{\infty} \frac{(\omega - \omega_0)^n}{n!} \text{Tr} \left(\frac{\partial^n W_L}{\partial \omega^n}(\beta, \omega_0) \right). \quad (2.12)$$

Proof of Theorem 2.3. We will use here some results from [HP] and [ABN2], which we briefly recall. Let $\omega_0 \geq 0$. For $\omega \in \mathbb{C}$ set $\delta\omega := \omega - \omega_0$. Then the operator

$$H_L(\omega) - H_L(\omega_0) = (\delta\omega) \mathbf{a} \cdot (i\nabla_{\mathbf{x}} + \omega_0 \mathbf{a}) + \frac{(\delta\omega)^2}{2} \mathbf{a}^2 \quad (2.13)$$

is relatively bounded to $H_L(\beta, \omega_0)$ with relative bound zero. For every compact subset $K \subset \mathbb{C}$, and due to the estimates (2.3), this operator satisfies

$$\int_0^1 d\tau \sup_{\omega \in K} \| (H_L(\omega) - H_L(\omega_0)) W_L(\tau, \omega_0) \| < \infty.$$

Let $0 < \beta_1 \leq \beta_0 < \infty$. Then the series

$$W_L(\beta, \omega, \omega_0) = \sum_{n=0}^{\infty} (-1)^n W_L^{(n)}(\beta, \omega, \omega_0) \quad (2.14)$$

where $W_L^{(0)}(\beta, \omega, \omega_0) = W_L(\beta, \omega_0)$ and for $n \geq 1$

$$W_L^{(n)}(\beta, \omega, \omega_0) = \int_0^\beta d\tau W_L(\beta - \tau, \omega_0) (H_L(\omega) - H_L(\omega_0)) W_L^{(n-1)}(\tau, \omega_0) \quad (2.15)$$

is uniformly B_1 -convergent on $K \times [\beta_1, \beta_0]$. Note that for $\beta > 0$ from (2.13) we have in the operator sense on \mathcal{H}_L

$$\begin{aligned} \hat{R}_L(\beta, \omega) &:= (H_L(\omega) - H_L(\omega_0)) W_L(\beta, \omega_0) \\ &= \delta\omega \hat{R}_{1,L}(\beta, \omega_0) + (\delta\omega)^2 \hat{R}_{2,L}(\beta, \omega_0). \end{aligned} \quad (2.16)$$

This result was obtained in [ABN2]. Since $W_L(\beta, \omega, \omega_0)$ is the uniform limit of a sequence of entire B_1 -valued functions, it follows via the Cauchy integral formula that $W_L(\beta, \omega, \omega_0)$ is also B_1 -entire in ω . Moreover, for real ω it coincides with the operator $e^{-\beta H_L(\omega)}$, $\beta > 0$.

What we do here is to identify its n th order derivative with respect to ω . From (2.15) and (2.16) a simple induction argument yields the following finite rearranging:

$$\begin{aligned} \sum_{n=0}^N (-1)^n W_L^{(n)}(\beta, \omega, \omega_0) &= W_L(\beta, \omega_0) + \sum_{n=1}^N (\delta\omega)^n \sum_{k=1}^n (-1)^k \sum_{i_j \in \{1,2\}} \chi_k^n(i_1, \dots, i_k) \\ &\cdot \hat{I}_{k,L}(i_1, \dots, i_k)(\beta, \omega_0) + \mathcal{R}_{N+1,L}(\beta, \omega, \omega_0), \end{aligned} \quad (2.17)$$

where

$$\begin{aligned} \mathcal{R}_{N+1,L}(\beta, \omega, \omega_0) & \\ &= \sum_{n=N+1}^{2N} (\delta\omega)^n \sum_{k=1}^N (-1)^k \sum_{i_j \in \{1,2\}} \chi_k^n(i_1, \dots, i_k) \hat{I}_{k,L}(i_1, \dots, i_k)(\beta, \omega_0). \end{aligned} \quad (2.18)$$

Now differentiation with respect to ω commutes with the limit $N \rightarrow \infty$, again due to the uniform convergence and the Cauchy integral formula. Hence (2.9) is proved, since the n th order derivative of $\sum_{j=0}^N (-1)^j W_L^{(j)}(\beta, \omega, \omega_0)$ at $\omega = \omega_0$ equals the right hand side of (2.10) if $N \geq n$.

In the second part of the proof, we use the methods of [ABN2] in order to estimate the B_1 -norm of the operators $I_{k,L}(i_1, \dots, i_k)$ as claimed in (2.11). We first have:

$$\begin{aligned} \|\hat{I}_{k,L}(i_1, \dots, i_k)(\beta, \omega_0)\|_1 &\leq \\ &\int_{\mathcal{D}_k(\beta)} d\tau \|W_L(\beta - \tau_1, \omega_0) \hat{R}_{i_1,L}(\tau_1 - \tau_2, \omega_0) \dots \hat{R}_{i_k,L}(\tau_k, \omega_0)\|_1. \end{aligned} \quad (2.19)$$

Recall that the Ginibre-Gruber inequality read as [ABN2],

$$\left\| \prod_{l=0}^k A_l T(t_l) \right\|_1 \leq \left(\prod_{l=0}^k \|A_l\| \right) \text{Tr } T(t_0 + t_1 + \dots + t_k) \quad (2.20)$$

where $\{A_l, 0 \leq l \leq k\}$ are bounded operators and $T(t), t > 0$ is a Gibbs semi-group. Then taking $A_0 := W_L(\frac{\beta-\tau_1}{2}, \omega_0)$, $A_l := \hat{R}_{i_l, L}(\frac{\tau_l - \tau_{l+1}}{2}, \omega_0)$ if $l \geq 1$ and $T(t) = W_L(\frac{t}{2}, \omega_0)$. On $D_k(\beta)$, we have the estimate $\|A_0\| \leq 1 \leq \sqrt{\frac{1+\beta}{\beta-\tau_1}}$ and by the Lemma 2.1 for $l \geq 1$

$$\|A_l\| = \|R_{i_l, L}(\frac{\tau_l - \tau_{l+1}}{2}, \omega_0)\| \leq \text{const} \cdot L^{i_l} \frac{\sqrt{1+\beta}}{\sqrt{\tau_l - \tau_{l+1}}}. \quad (2.21)$$

Let $f_k : \mathcal{D}_n(\beta) \rightarrow \mathbb{R}$ defined as

$$f_k(\tau) := \frac{1}{\sqrt{(\beta - \tau_1)(\tau_1 - \tau_2) \dots (\tau_{k-1} - \tau_k) \tau_k}}, \quad (2.22)$$

and it satisfies

$$\int_{\mathcal{D}_k(\beta)} f_k(\tau) d\tau = \frac{\beta^{\frac{k-1}{2}} \pi^{\frac{k+1}{2}}}{\Gamma(\frac{k+1}{2})}. \quad (2.23)$$

Let $i_1 + \dots + i_k = n$. Then from (2.5), (2.20), (2.21) and (2.23), we obtain the existence of a numerical constant C , such that for every $\beta > 0$:

$$\|\hat{I}_{k, L}(i_1, \dots, i_k)(\beta, \omega_0)\|_1 \leq \frac{L^{n+3} C^k (1+\beta)^k}{\beta^{3/2} \Gamma(\frac{k+1}{2})}. \quad (2.24)$$

Then we have the estimate (see (2.10)):

$$\frac{1}{n!} \|(\partial_\omega^n W_L)(\beta, \omega_0)\|_1 \leq C^n L^{n+3} \frac{(1+\beta)^n}{\beta^{3/2}} \sum_{k=1}^n \sum_{i_j \in \{1,2\}} \frac{\chi_k^n(i_1, \dots, i_k)}{\Gamma(\frac{k+1}{2})}. \quad (2.25)$$

But a lot of terms in the above sum are zero, since $\chi_k^n(i_1, \dots, i_k) = 0$ if $k < [\frac{n+1}{2}]$. Since Γ is increasing, we can give a rough estimate of the form:

$$\sum_{k=1}^n \sum_{i_j \in \{1,2\}} \frac{\chi_k^n(i_1, \dots, i_k)}{\Gamma(\frac{k+1}{2})} \leq n 2^n \frac{1}{\Gamma(\frac{[(n+1)/2]+1}{2})} \leq n 2^n \frac{1}{[(n-1)/4]!}. \quad (2.26)$$

□

2.2 Proof of Theorem 1.1.

The analyticity properties of the pressure are now easy to prove once we have the B_1 -analyticity of the Gibbs semigroup. See [ABN1] for details.

Now let $\beta > 0, \omega \geq 0$ and $|z| < 1$. Since $\|z W_L(\beta, \omega)\|_\infty < 1$, the logarithm in the pressure at finite volume can be expanded and we obtain:

$$P_L(\beta, \omega, z, \epsilon) = \epsilon / (\beta |\Lambda_L|) \sum_{k \geq 1} (-\epsilon z)^k / k \text{Tr } W_L(k\beta, \omega). \quad (2.27)$$

Starting from Definition (1.5), and using Theorem 2.3 we obtain:

$$\chi_L^{(n)}(\beta, \omega, z, \epsilon) = \epsilon / (\beta |\Lambda_L|) \sum_{k \geq 0} (-\epsilon z)^k / k \text{Tr} \left(\frac{\partial^n W_L(k\beta, \omega)}{\partial \omega^n} \right). \quad (2.28)$$

Note that (2.11) insures that the growth in k which comes from the trace of the n th derivative of $W_L(k\beta, \omega)$ is not faster than some polynomial, but since $|z| < 1$ the series in k is convergent. This finishes the proof of the theorem. □

2.3 Analyticity of the semigroup's integral kernel

In the rest of this paper we will only consider $\Lambda_L = (-L/2, L/2)^3, L \geq 1$. For $\omega \in \mathbb{R}$, $H_L(\omega)$ is essentially selfadjoint on

$$\{\phi \in C^1(\bar{\Lambda}_L) \cup C^2(\Lambda_L), \phi|_{\partial\Lambda_L} = 0, \Delta\phi \in L^2(\Lambda_L)\}.$$

Let $G_L(\mathbf{x}, \mathbf{x}'; \beta, \omega)$ be the integral kernel of $W_L(\beta, \omega)$ (see e.g. [AC]). Standard elliptic estimates for the eigenfunctions of $H_L(\omega)$, together with the fact that $e^{-\beta H_L(\omega)}$ is trace class imply that $G_L(\mathbf{x}, \mathbf{x}'; \beta, \omega)$ is smooth in $(\mathbf{x}, \mathbf{x}') \in \Lambda_L \times \Lambda_L$. Moreover, $G_L(\mathbf{x}, \mathbf{x}'; \beta, \omega) = 0$ if either \mathbf{x} or \mathbf{x}' are on the boundary.

To prove the next theorem, we need the following result from [C], concerning the C^1 regularity up to the boundary of the integral kernel. Let $\beta > 0$ and let $G_\infty(\mathbf{x}, \mathbf{x}', \beta) := G_\infty(\mathbf{x}, \mathbf{x}', \beta, \omega = 0)$ be the heat kernel on the whole space, i.e.

$$G_\infty(\mathbf{x}, \mathbf{x}'; \beta) = \frac{1}{(2\pi\beta)^{3/2}} e^{-\frac{|\mathbf{x}-\mathbf{x}'|^2}{2\beta}}. \quad (2.29)$$

Recall that the diamagnetic estimate reads as [AC]:

$$|G_L(\mathbf{x}, \mathbf{x}'; \beta, \omega)| \leq G_\infty(\mathbf{x}, \mathbf{x}'; \beta), \quad (\mathbf{x}, \mathbf{x}') \in \Lambda_L \times \Lambda_L, \quad \omega \geq 0. \quad (2.30)$$

Then we have

Lemma 2.5. *Let $\beta > 0$ and $\omega \geq 0$. Then on $\Lambda_L \times \Lambda_L$ we have:*

$$|(i\nabla_{\mathbf{x}} + \omega \mathbf{a}(\mathbf{x}))G_L(\mathbf{x}, \mathbf{x}'; \beta, \omega)| \leq \frac{\mathcal{C}}{\sqrt{\beta}} G_\infty(\mathbf{x}, \mathbf{x}'; 8\beta). \quad (2.31)$$

where $\mathcal{C} = \mathcal{C}(\beta, \omega) = c \cdot (1 + \beta)^5 (1 + \omega)^3$ and $c > 1$ is a numerical constant.

This estimate allows us to define the integral kernels of the operators defined in (2.2); more precisely, for $(\mathbf{x}, \mathbf{x}') \in \Lambda_L \times \Lambda_L$ we have:

$$\begin{aligned} \hat{R}_{1,L}(\mathbf{x}, \mathbf{x}'; \beta, \omega) &:= \mathbf{a}(\mathbf{x}) \cdot (i\nabla_{\mathbf{x}} + \omega \mathbf{a}(\mathbf{x}))G_L(\mathbf{x}, \mathbf{x}'; \beta, \omega), \\ \hat{R}_{2,L}(\mathbf{x}, \mathbf{x}'; \beta, \omega) &:= \frac{1}{2} \mathbf{a}^2(\mathbf{x}) G_L(\mathbf{x}, \mathbf{x}'; \beta, \omega). \end{aligned} \quad (2.32)$$

Consider the operator $W_L(\beta, \omega)$ for complex ω , defined by a B_1 -convergent complex power series in Theorem 2.3. We will now prove that it has an integral kernel analytic in ω :

Theorem 2.6. *Let $\beta > 0$ and fix $\omega_0 \geq 0$.*

(i). *The operator $(\partial_\omega^n W_L)(\beta, \omega_0)$ defined in (2.10) has an integral kernel denoted by $(\partial_\omega^n W_L)(\mathbf{x}, \mathbf{x}'; \beta, \omega_0)$, which is jointly continuous on $(\mathbf{x}, \mathbf{x}') \in \bar{\Lambda}_L \times \bar{\Lambda}_L$, and obeys the estimate*

$$\frac{1}{n!} \left| \frac{\partial^n W_L}{\partial \omega^n}(\mathbf{x}, \mathbf{x}'; \beta, \omega_0) \right| \leq c^n \frac{(1 + \omega_0)^{3n} (1 + \beta)^{6n} L^n}{\beta^{3/2} \left[\frac{n-1}{4}\right]!} \quad n \geq 1 \quad (2.33)$$

for some numerical constant $c \geq 1$.

(ii). *For $\omega \in \mathbb{C}$, the operator $W_L(\beta, \omega)$ has an integral kernel $G_L(\mathbf{x}, \mathbf{x}'; \beta, \omega)$ given by:*

$$G_L(\mathbf{x}, \mathbf{x}'; \beta, \omega) = \sum_{n=0}^{\infty} \frac{(\omega - \omega_0)^n}{n!} \left(\frac{\partial^n W_L}{\partial \omega^n} \right)(\mathbf{x}, \mathbf{x}'; \beta, \omega_0), \quad (2.34)$$

where the above series is uniformly convergent on $\bar{\Lambda}_L \times \bar{\Lambda}_L$. Thus G_L is jointly continuous on $\bar{\Lambda}_L \times \bar{\Lambda}_L$ and is an entire function of ω .

Proof of the Theorem 2.6. Lemma 2.5 obviously implies for $\beta > 0$ and $\omega \geq 0$ the estimate:

$$|\hat{R}_{1,L}(\mathbf{x}, \mathbf{x}'; \beta, \omega)| \leq L \frac{\mathcal{C}}{\sqrt{\beta}} G_\infty(\mathbf{x}, \mathbf{x}'; 8\beta). \quad (2.35)$$

We also have

$$|\hat{R}_{2,L}(\mathbf{x}, \mathbf{x}'; \beta, \omega)| \leq \frac{L^2}{4} G_\infty(\mathbf{x}, \mathbf{x}'; \beta). \quad (2.36)$$

In the following, we will often use the uniform estimate with respect to the index $i = 1, 2$:

$$|\hat{R}_{i,L}(\mathbf{x}, \mathbf{x}'; t, \omega)| \leq L^i \mathcal{C}_1 \sqrt{\frac{(1+\beta)}{t}} G_\infty(\mathbf{x}, \mathbf{x}'; 8t) \quad 0 < t \leq \beta, \quad (2.37)$$

where $\mathcal{C}_1 := \mathcal{C}_1(\beta, \omega) = 2\sqrt{2}\mathcal{C}(\beta, \omega)$.

Let us start by proving (i). Fix $\beta > 0, \omega_0 \geq 0, L \geq 1$, and consider the operator $\hat{I}_{k,L}(i_1, \dots, i_k)(\beta, \omega_0)$, $k \geq 1$ defined in (2.7). It admits a continuous integral kernel, $\hat{I}_{k,L}(\mathbf{x}, \mathbf{x}'; \beta, \omega_0) := \hat{I}_{k,L}(i_1, \dots, i_k)(\mathbf{x}, \mathbf{x}'; \beta, \omega_0)$ on $\Lambda_L \times \Lambda_L$ given by

$$\begin{aligned} & \hat{I}_{k,L}(\mathbf{x}, \mathbf{x}'; \beta, \omega_0) \\ &= \int_{D_k(\beta)} d\tau \int_{\Lambda_L^k} d\mathbf{y} G_L(\mathbf{x}, \mathbf{y}_1; \beta - \tau_1, \omega_0) \hat{R}_{i_1,L}(\mathbf{y}_1, \mathbf{y}_2; \tau_1 - \tau_2, \omega_0) \\ & \quad \cdots \hat{R}_{i_{k-1},L}(\mathbf{y}_{k-1}, \mathbf{y}_n; \tau_{k-1} - \tau_k, \omega_0) \hat{R}_{i_k,L}(\mathbf{y}_k, \mathbf{x}'; \tau_k, \omega_0) \end{aligned} \quad (2.38)$$

where $d\mathbf{y}$ denotes the Lebesgue measure on \mathbb{R}^{3k} and $D_k(\beta)$ is defined in (2.6). Let $i_1 + i_2 + \dots + i_k = n$. Then by using the Lemma 2.5, the estimate

$$|G_L(\mathbf{x}, \mathbf{y}_1; \beta - \tau_1, \omega_0)| \leq 8^{3/2} \sqrt{\frac{\beta}{\beta - \tau_1}} G_\infty(\mathbf{x}, \mathbf{y}_1; 8(\beta - \tau_1)), \quad 0 < \tau_1 < \beta,$$

and (2.37), the following estimate holds on $\Lambda_L \times \Lambda_L$:

$$\begin{aligned} |\hat{I}_{k,L}(\mathbf{x}, \mathbf{x}'; \beta, \omega_0)| &\leq 8^{3/2} (1+\beta)^{(k+1)/2} \mathcal{C}_1^k L^n \int_{D_k(\beta)} f_k(\tau) d\tau \int_{\Lambda_L^k} d\mathbf{y} \\ & \quad G_\infty(\mathbf{x}, \mathbf{y}_1; 8(\beta - \tau_1)) \dots G_\infty(\mathbf{y}_k, \mathbf{x}'; 8\tau_k), \end{aligned} \quad (2.39)$$

where the function f_k is defined in (2.22). Notice that by using the semigroup property

$$\begin{aligned} \int_{\Lambda_L^k} d\mathbf{y} G_\infty(\mathbf{x}, \mathbf{y}_1; t_1) \dots G_\infty(\mathbf{y}_k, \mathbf{x}'; t_k) &\leq \int_{\mathbb{R}^k} d\mathbf{y} G_\infty(\mathbf{x}, \mathbf{y}_1; t_1) \dots G_\infty(\mathbf{y}_k, \mathbf{x}'; t_k) \\ &= G_\infty(\mathbf{x}, \mathbf{x}'; t_1 + \dots + t_k) \end{aligned} \quad (2.40)$$

Therefore from (2.39) we get

$$|\hat{I}_{k,L}(\mathbf{x}, \mathbf{x}'; \beta, \omega_0)| \leq 8^{3/2} (1+\beta)^{(k+1)/2} \mathcal{C}_1^k L^n G_\infty(\mathbf{x}, \mathbf{x}'; 8\beta) \int_{D_k(\beta)} f_k(\tau) d\tau. \quad (2.41)$$

Then Theorem 2.3 together with (2.23) and (2.41) show that the operator $\frac{\partial^n W_L}{\partial \omega^n}(\beta, \omega_0), n \geq 1$ given by (2.9), admits a continuous integral kernel satisfying

$$\left| \frac{\partial^n W_L}{\partial \omega^n}(\mathbf{x}, \mathbf{x}'; \beta, \omega_0) \right| \leq (8^3 \pi)^{1/2} \mathcal{C}_2^n n! L^n G_\infty(\mathbf{x}, \mathbf{x}'; 8\beta) \sum_{k=1}^n \sum_{i_j \in \{1,2\}} \frac{\chi_k^n(i_1, \dots, i_k)}{\Gamma\left(\frac{k+1}{2}\right)}.$$

for a new constant $\mathcal{C}_2 = \mathcal{C}_2(\beta, \omega_0) := \pi^{1/2}(1 + \beta)\mathcal{C}_1(\beta, \omega_0)$. Then by mimicking the proof of (2.26) we get from the last inequality

$$\frac{1}{n!} \left| \frac{\partial^n W_L}{\partial \omega^n}(\mathbf{x}, \mathbf{x}'; \beta, \omega_0) \right| \leq \frac{c^n (1 + \beta)^{6n} (1 + \omega_0)^{3n} L^n}{[(n-1)/4]!} G_\infty(\mathbf{x}, \mathbf{x}'; 8\beta), \quad (2.42)$$

where c is a positive numerical constant. Since $G_\infty(\mathbf{x}, \mathbf{x}'; 8\beta) \leq (2\pi\beta)^{-3/2}$, (2.42) implies the estimate (2.33) and proves (i). Then (ii) follows easily from the previous estimate since $1/[(n-1)/4]!$ has a super-exponential decay in n , which is seen for example with the Stirling approximation formula. \square

3 Regularized expansion

The bounds obtained in the previous section are not convenient for the proof of the existence of the thermodynamic limit of the magnetic susceptibilities. In particular the bound on $\frac{\partial^n W_L}{\partial \omega^n}(\mathbf{x}, \mathbf{x}', \beta, \omega_0)$ given by (2.42) is of order L^n . Then this gives a bound on its trace of order L^{3+n} , while in view of (1.6) we would like to have L^3 .

In this section, we give an improvement of these estimates. In order to do that we need to introduce the magnetic phase ϕ and the magnetic flux fl defined as (here $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \Lambda_L$ and $\mathbf{e} = (0, 0, 1)$):

$$\phi(\mathbf{x}, \mathbf{y}) := \frac{1}{2} \mathbf{e} \cdot (\mathbf{y} \wedge \mathbf{x}) = -\phi(\mathbf{y}, \mathbf{x}), \quad (3.1)$$

$$fl(\mathbf{x}, \mathbf{y}, \mathbf{z}) := \phi(\mathbf{x}, \mathbf{y}) + \phi(\mathbf{y}, \mathbf{z}) + \phi(\mathbf{z}, \mathbf{x}) = \frac{1}{2} \mathbf{e} \cdot \{(\mathbf{x} - \mathbf{y}) \wedge (\mathbf{z} - \mathbf{y})\}. \quad (3.2)$$

Note that fl is really the magnetic flux through the triangle defined by the three vectors, and we have:

$$|fl(\mathbf{x}, \mathbf{y}, \mathbf{z})| \leq |\mathbf{x} - \mathbf{y}| |\mathbf{y} - \mathbf{z}|. \quad (3.3)$$

For $n \geq 1$ and $\mathbf{x} = \mathbf{y}_0, \mathbf{y}_1, \dots, \mathbf{y}_n$ some arbitrary vectors in Λ_L , define

$$\begin{aligned} Fl_n(\mathbf{x}, \mathbf{y}_1, \dots, \mathbf{y}_n) &:= \sum_{k=0}^{n-1} \phi(\mathbf{y}_k, \mathbf{y}_{k+1}) = \sum_{k=1}^{n-1} fl(\mathbf{x}, \mathbf{y}_k, \mathbf{y}_{k+1}), \text{ if } n \geq 2 \\ \text{and } Fl_1(\mathbf{x}, \mathbf{y}_1) &= 0. \end{aligned} \quad (3.4)$$

Notice that due to (3.3), we have

$$|Fl_n(\mathbf{x}, \mathbf{y}_1, \dots, \mathbf{y}_n)| \leq \sum_{k=1}^{n-1} \sum_{l=1}^k |\mathbf{y}_{l-1} - \mathbf{y}_l| |\mathbf{y}_k - \mathbf{y}_{k+1}|. \quad (3.5)$$

Let $\omega \geq 0$. Consider now the bounded operators given by their integral kernels on $\Lambda_L \times \Lambda_L$,

$$\begin{aligned} R_{1,L}(\mathbf{x}, \mathbf{x}'; \beta, \omega) &:= \mathbf{a}(\mathbf{x} - \mathbf{x}') \cdot (i\nabla_{\mathbf{x}} + \omega \mathbf{a}(\mathbf{x})) G_L(\mathbf{x}, \mathbf{x}'; \beta, \omega). \\ R_{2,L}(\mathbf{x}, \mathbf{x}'; \beta, \omega) &:= \frac{1}{2} \mathbf{a}^2(\mathbf{x} - \mathbf{x}') G_L(\mathbf{x}, \mathbf{x}'; \beta, \omega). \end{aligned} \quad (3.6)$$

Then by the Lemma 2.5, a straightforward estimate yields to

$$|R_{1,L}(\mathbf{x}, \mathbf{x}'; \beta, \omega)| \leq \frac{\mathcal{C}|\mathbf{x} - \mathbf{x}'|}{2\sqrt{\beta}} G_{\infty}(\mathbf{x}, \mathbf{x}', 8\beta) \leq 4\mathcal{C}_1 G_{\infty}(\mathbf{x}, \mathbf{x}', 16\beta) \quad (3.7)$$

for all $(\mathbf{x}, \mathbf{x}') \in \Lambda_L \times \Lambda_L$. Similarly by (2.30) we have

$$|R_{2,L}(\mathbf{x}, \mathbf{x}'; \beta, \omega)| \leq \frac{|\mathbf{x} - \mathbf{x}'|^2}{8} G_{\infty}(\mathbf{x}, \mathbf{x}', \beta) \leq \frac{\beta}{\sqrt{2}} G_{\infty}(\mathbf{x}, \mathbf{x}', 2\beta). \quad (3.8)$$

In the sequel for $i = 1, 2$ we will use the estimate on $\Lambda_L \times \Lambda_L$

$$|R_{i,L}(\mathbf{x}, \mathbf{x}'; \beta, \omega)| \leq \mathcal{C}_3 G_{\infty}(\mathbf{x}, \mathbf{x}', 16\beta) \quad (3.9)$$

where $\mathcal{C}_3 = \mathcal{C}_3(\beta, \omega) := 16\mathcal{C}_1(\beta, \omega)$ and \mathcal{C}_1 is given in (2.37).

Notice that (3.9) provides an uniform bound w.r.t. L and β near $\beta = 0$ on the operator kernels. This in contrast with the bound on the norm operator of $\hat{R}_{i,L}; i = 1, 2$ (see section 2.2, (2.35) and (2.36)). Using the Schur-Holmgren estimate for the operator norm of an integral operator, (3.9) eventually implies

$$\|R_{i,L}\| \leq \mathcal{C}_3, \quad i = 1, 2. \quad (3.10)$$

Let $\mathbf{x} \in \Lambda_L$. For $k \geq 1, m \geq 0, \omega \geq 0$ and $\beta > 0$, define the continuous function

$$\begin{aligned} W_{k,L}^m(\mathbf{x}; \beta, \omega) &:= \sum_{j=1}^k (-1)^j \sum_{(i_1, \dots, i_j) \in \{1,2\}^j} \chi_j^k(i_1, \dots, i_j) \int_{D_j(\beta)} d\tau \int_{\Lambda^j} d\mathbf{y} \\ &\frac{(i(\text{Fl}_j(\mathbf{x}, \mathbf{y}_1, \dots, \mathbf{y}_j)))^m}{m!} G_L(\mathbf{x}, \mathbf{y}_1; \beta - \tau_1, \omega) R_{i_1,L}(\mathbf{y}_1, \mathbf{y}_2; \tau_1 - \tau_2, \omega) \\ &\dots R_{i_{j-1},L}(\mathbf{y}_{j-1}, \mathbf{y}_j; \tau_{j-1} - \tau_j, \omega) R_{i_j,L}(\mathbf{y}_j, \mathbf{x}; \tau_j, \omega), \end{aligned} \quad (3.11)$$

where in the case of $m = 0$ we set $0^0 \equiv 1$.

The main result of this section gives a new expression for the diagonal of kernel's n th derivative with respect to ω at finite volume.

Theorem 3.1. *Let $\beta > 0$ and $\omega_0 \geq 0$. Then for all $\mathbf{x} \in \Lambda$, and for all $n \geq 1$, one has*

$$\frac{1}{n!} \frac{\partial^n G_L}{\partial \omega^n}(\mathbf{x}, \mathbf{x}, \beta, \omega_0) = \sum_{k=1}^n W_{k,L}^{n-k}(\mathbf{x}; \beta, \omega_0). \quad (3.12)$$

Proof. We first need the following result. Fix $\omega_0 \geq 0$. Let $\omega \in \mathbb{C}$, $\delta\omega = \omega - \omega_0$ and $\tilde{R}_{i,L}(\beta, \omega, \omega_0), i = 1, 2, \tilde{W}_L(\beta, \omega, \omega_0)$ be the operators on \mathcal{H}_L defined via their respective integral kernel given on $\Lambda_L \times \Lambda_L$, by

$$\begin{aligned} \tilde{R}_{i,L}(\mathbf{x}, \mathbf{x}'; \beta, \omega, \omega_0) &= e^{i\delta\omega\phi(\mathbf{x}, \mathbf{x}')} R_{i,L}(\mathbf{x}, \mathbf{x}'; \beta, \omega_0), \quad i = 1, 2, \\ \tilde{W}_L(\mathbf{x}, \mathbf{x}'; \beta, \omega, \omega_0) &= e^{i\delta\omega\phi(\mathbf{x}, \mathbf{x}')} G_L(\mathbf{x}, \mathbf{x}'; \beta, \omega_0) \end{aligned} \quad (3.13)$$

where ϕ is defined in (3.1). We also set

$$\tilde{R}_L(\mathbf{x}, \mathbf{x}'; \beta, \omega, \omega_0) := \delta\omega \tilde{R}_{1,L}(\mathbf{x}, \mathbf{x}'; \beta, \omega, \omega_0) + (\delta\omega)^2 \tilde{R}_{2,L}(\mathbf{x}, \mathbf{x}'; \beta, \omega, \omega_0).$$

Except a phase factor the kernel of \tilde{W}_L and $\tilde{R}_{i,L}$, $i = 1, 2$ is the same as the one of W_L , $R_{i,L}$, $i = 1, 2$ respectively. Then they satisfy (2.30), (3.9) respectively. Hence by the same arguments as above, they are bounded operators and

$$\|\tilde{W}_L\| \leq 1, \|\tilde{R}_{i,L}\| \leq \mathcal{C}_3 \quad (3.14)$$

(see (3.10). Notice also that since $\|\tilde{W}_L\|_{\mathcal{I}_2} = \|W_L\|_{\mathcal{I}_2} \leq \frac{L^{3/2}}{(2\pi\beta)^{3/4}}$ then by (3.9) $R_{i,L}$, $i = 1, 2$ as well as $\tilde{R}_{i,L}$, $i = 1, 2$ are in the Hilbert-Schmidt class and for $\beta > 0$, $\omega_0 \geq 0$ and $\omega \in \mathbb{C}$,

$$\|R_{i,L}(\beta, \omega)\|_{\mathcal{I}_2}, \|\tilde{R}_{i,L}(\beta, \omega, \omega_0)\|_{\mathcal{I}_2} \leq \mathcal{C}_3 \|W_L(16\beta, \omega)\|_{\mathcal{I}_2} \leq \mathcal{C}_3 \frac{L^{3/2}}{(2\pi\beta)^{3/4}}. \quad (3.15)$$

where $\mathcal{C}_3 = \mathcal{C}_3(\beta, \omega_0)$. We now define the following family of bounded operators on \mathcal{H}_L . Let $k \geq 1$, $\{i_1, \dots, i_k\} \in \{1, 2\}^k$, $\beta > 0$, $\omega_0 \geq 0$. For all $\omega \in \mathbb{C}$ set

$$\begin{aligned} \tilde{I}_{k,L}(i_1, \dots, i_k)(\beta, \omega, \omega_0) &:= \int_{D_k(\beta)} d\tau \tilde{W}_L(\beta - \tau_1, \omega, \omega_0) \tilde{R}_{i_1,L}(\tau_1 - \tau_2, \omega, \omega_0) \\ &\dots \tilde{R}_{i_{k-1},L}(\tau_{k-1} - \tau_k, \omega, \omega_0) \tilde{R}_{i_k,L}(\tau_k, \omega, \omega_0), \end{aligned} \quad (3.16)$$

and for $n \geq 1$

$$W_{n,L}(\beta, \omega, \omega_0) := \sum_{k=1}^n (-1)^k \sum_{i_j \in \{1,2\}} \chi_k^n(i_1, \dots, i_k) \tilde{I}_{k,L}(i_1, \dots, i_k)(\beta, \omega, \omega_0) \quad (3.17)$$

where χ_k^n is defined in (2.8).

Lemma 3.2. *Let $N \geq 1$, $\beta > 0$, $\omega_0 \geq 0$. For all $\omega \in \mathbb{C}$ set $\delta\omega = \omega - \omega_0$. Then as bounded operators we can write:*

$$\begin{aligned} W_L(\beta, \omega) &= \tilde{W}_L(\beta, \omega, \omega_0) + \sum_{n=1}^N (\delta\omega)^n W_{n,L}(\beta, \omega, \omega_0) + \\ &\tilde{\mathcal{R}}_{N+1,L}^{(1)}(\beta, \omega, \omega_0) + \tilde{\mathcal{R}}_{N+1,L}^{(2)}(\beta, \omega, \omega_0), \end{aligned} \quad (3.18)$$

where $\tilde{\mathcal{R}}_{N+1,L}^{(1)}(\beta, \omega, \omega_0)$ and $\tilde{\mathcal{R}}_{N+1,L}^{(2)}(\beta, \omega, \omega_0)$ are the following bounded operators on \mathcal{H}_L :

$$\begin{aligned} \tilde{\mathcal{R}}_{N+1,L}^{(1)}(\beta, \omega, \omega_0) &:= (-1)^{N+1} \sum_{n=N+1}^{2N+2} (\delta\omega)^n \sum_{i_j \in \{1,2\}} \chi_{N+1}^n(i_1, \dots, i_n) \\ &\cdot \int_{D_{N+1}(\beta)} d\tau W_L(\beta - \tau_1, \omega) \tilde{\mathcal{R}}_{i_1,L}(\tau_1 - \tau_2, \omega, \omega_0) \dots \tilde{\mathcal{R}}_{i_{N+1},L}(\tau_{N+1}, \omega, \omega_0) \end{aligned} \quad (3.19)$$

where $D_{N+1}(\beta)$ is given in (2.6) and

$$\begin{aligned} &\tilde{\mathcal{R}}_{N+1,L}^{(2)}(\beta, \omega, \omega_0) \\ &= \sum_{n=N+1}^{2N} (\delta\omega)^n \sum_{k=1}^N (-1)^k \sum_{i_j \in \{1,2\}} \chi_k^n(i_1, \dots, i_k) \cdot \tilde{I}_{k,L}(i_1, \dots, i_k)(\beta, \omega, \omega_0). \end{aligned} \quad (3.20)$$

Proof of the lemma. In this proof we fix $\omega_0 \geq 0$ and omit everywhere the ω_0 dependence. We first note that $\widetilde{W}_L(\beta, \omega)$ is strongly differentiable with respect to $\beta \geq 0$ on $\text{Dom}(H_L(\omega))$ (see [C]) and satisfies

$$\frac{\partial \widetilde{W}_L(\beta, \omega)}{\partial \beta} + H_L(\omega) \widetilde{W}_L(\beta, \omega) = \widetilde{R}_L(\beta, \omega).$$

By using a result from [C], we can write the following integral equation which holds in the bounded operators sense on \mathcal{H}_L :

$$W_L(\beta, \omega) = \widetilde{W}_L(\beta, \omega) - \int_0^\beta d\tau W_L(\beta - \tau, \omega) \widetilde{R}_L(\tau, \omega). \quad (3.21)$$

By iterating (3.21) we obtain:

$$\begin{aligned} W_L(\beta, \omega) &= \widetilde{W}_L(\beta, \omega) + \sum_{n=1}^N (-1)^n \int_0^\beta d\tau_1 \int_0^{\beta-\tau_1} d\tau_2 \dots \int_0^{\beta-\tau_1-\dots-\tau_{n-1}} d\tau_n \\ &\cdot \widetilde{W}_L(\beta - \tau_1 - \dots - \tau_n, \omega) \widetilde{R}_L(\tau_n, \omega) \dots \widetilde{R}_L(\tau_1, \omega) + \widetilde{\mathcal{R}}_{N+1,L}^{(1)}(\beta, \omega) \end{aligned} \quad (3.22)$$

where

$$\begin{aligned} \widetilde{\mathcal{R}}_{N+1,L}^{(1)}(\beta, \omega) &= (-1)^{N+1} \int_0^\beta d\tau_1 \int_0^{\beta-\tau_1} d\tau_2 \dots \int_0^{\beta-\tau_1-\dots-\tau_N} d\tau_{N+1} \\ &\cdot W_L(\beta - \tau_1 - \dots - \tau_{N+1}, \omega) \cdot \widetilde{R}_L(\tau_{N+1}, \omega) \dots \widetilde{R}_L(\tau_1, \omega). \end{aligned} \quad (3.23)$$

Then a straightforward change of variables in the integrals of the r.h.s. of the last two formulas, yields to

$$\begin{aligned} W_L(\beta, \omega) &= \widetilde{W}_L(\beta, \omega) + \sum_{n=1}^N (-1)^n \int_{D_n(\beta)} d\tau \widetilde{W}_L(\beta - \tau_1, \omega) \widetilde{R}_L(\tau_1 - \tau_2, \omega) \\ &\dots \widetilde{R}_L(\tau_{n-1} - \tau_n, \omega) \widetilde{R}_L(\tau_n, \omega) + \widetilde{\mathcal{R}}_{N+1,L}^{(1)}(\beta, \omega). \end{aligned}$$

with

$$\widetilde{\mathcal{R}}_{N+1,L}^{(1)}(\beta, \omega) = (-1)^{N+1} \int_{D_{N+1}(\beta)} d\tau W_L(\beta - \tau_1, \omega) \cdot \widetilde{R}_L(\tau_1, \omega) \dots \widetilde{R}_L(\tau_{N+1}, \omega) \quad (3.24)$$

where $D_n(\beta)$ is defined in (2.6). Recall that $\widetilde{R}_L = \delta\omega \widetilde{R}_{1,L} + (\delta\omega)^2 \widetilde{R}_{2,L}$. So (3.24) gives (3.19) and a simple induction argument finishes the proof of the lemma. \square

Remark 3.3. Let $\beta > 0$ and $\omega_0 \geq 0$ and $\omega \in \mathbb{C}$. From (3.15) we have

$$\begin{aligned} \|W_L(\beta - \tau, \omega, \omega_0) \widetilde{R}_L(\tau, \omega, \omega_0)\|_1 &\leq \|W_L(\beta - \tau, \omega, \omega_0)\|_2 \|\widetilde{R}_L(\tau, \omega, \omega_0)\|_2 \\ &\leq \frac{C(\beta, \omega, \omega_0) L^3}{(\beta - \tau)^{3/4} \tau^{3/4}}, \end{aligned}$$

where $C(\beta, \omega, \omega_0) = \mathcal{C}_3(\beta, \omega_0)(|\delta\omega| + |(\delta\omega)|^2)$ and $\mathcal{C}_3(\beta, \omega_0)$ is given in (3.9). Then the B_1 -operator valued function $\tau \in (0, \beta) \rightarrow W_L(\beta - \tau, \omega)\tilde{R}_L(\tau, \omega, \omega_0)$ is B_1 -integrable. Denote by $U_L(\beta, \omega, \omega_0) = \int_0^\beta W_L(\beta - \tau, \omega)\tilde{R}_L(\tau, \omega, \omega_0)$,

$$\|U_L(\beta, \omega, \omega_0)\|_1 \leq CL^3 \int_0^\beta \frac{1}{(\beta - \tau)^{3/4}\tau^{3/4}} \leq \frac{16CL^3}{\sqrt{\beta}}. \quad (3.25)$$

The Duhamel-type formula (3.21) then implies that $\widetilde{W}_L(\beta, \omega)$ is of trace class as a sum of the two trace class operators, W_L and U_L . Consequently the operators $\tilde{I}_{k,L}(i_1, \dots, i_k)(\beta, \omega, \omega_0)$ defined in (3.16) are of trace class because all singularities are integrable.

Proof of the Theorem 3.1. From Theorem 2.6, we know that for $\mathbf{x} \in \Lambda_L$, and $\beta > 0$, $\mathbb{C} \ni \omega \rightarrow G_L(\mathbf{x}, \mathbf{x}; \beta, \omega)$ is an entire function.

In order to prove Theorem 3.1 we will show that there exists a positive constant $C(N) := C(N, \beta, \omega_0)$ such that for $|\delta\omega|$ small enough and all $\mathbf{x} \in \Lambda_L$ we have:

$$\begin{aligned} G_L(\mathbf{x}, \mathbf{x}; \beta, \omega) &= G_L(\mathbf{x}, \mathbf{x}; \beta, \omega_0) + \sum_{n=1}^N (\delta\omega)^n \sum_{k=1}^n W_{k,L}^{n-k}(\mathbf{x}; \beta, \omega_0) \\ &\quad + \tilde{\mathcal{R}}_{N+1,L}(\mathbf{x}; \beta, \omega, \omega_0) \end{aligned} \quad (3.26)$$

where the remainder term satisfies $|\tilde{\mathcal{R}}_{N+1,L}(\mathbf{x}; \beta, \omega, \omega_0)| \leq C(N)|\delta\omega|^{N+1}$.

By rewriting Lemma 3.2 in terms of the corresponding integral kernels, and looking at the diagonal of these kernels, we have:

$$\begin{aligned} G_L(\mathbf{x}, \mathbf{x}; \beta, \omega) &= G_L(\mathbf{x}, \mathbf{x}; \beta, \omega_0) + \sum_{n=1}^N (\delta\omega)^n W_{n,L}(\mathbf{x}, \mathbf{x}; \beta, \omega, \omega_0) \\ &\quad + \tilde{\mathcal{R}}_{N+1,L}^{(1)}(\mathbf{x}, \mathbf{x}; \beta, \omega, \omega_0) + \tilde{\mathcal{R}}_{N+1,L}^{(2)}(\mathbf{x}, \mathbf{x}; \beta, \omega, \omega_0). \end{aligned} \quad (3.27)$$

Let us show that for $1 \leq n \leq N$, $\mathbf{x} \in \Lambda_L$ and $|\delta\omega| < 1$,

$$W_{n,L}(\mathbf{x}, \mathbf{x}; \beta, \omega, \omega_0) = \sum_{m=0}^N (\delta\omega)^m W_{n,L}^m(\mathbf{x}; \beta, \omega_0) + \tilde{\mathcal{R}}_{n,N+1}^{(3)}(\mathbf{x}; \beta, \omega, \omega_0), \quad (3.28)$$

where $W_{n,L}^m$ were introduced in (3.11), and $\tilde{\mathcal{R}}_{n,N+1}^{(3)}(\mathbf{x}; \beta, \omega, \omega_0)$ has its first N derivatives at ω_0 equal to 0. Indeed, if we replace the integral kernel of $\tilde{I}_{k,L}$ from (3.16) in the expression of $W_{N,L}$ from (3.17), we see that we can add up all the magnetic phases, and obtain a factor of the type:

$$\exp\{\phi(\mathbf{x}, \mathbf{y}_1) + \phi(\mathbf{y}_1, \mathbf{y}_2) + \dots + \phi(\mathbf{y}_{k-1}, \mathbf{y}_k) + \phi(\mathbf{y}_k, \mathbf{x})\}.$$

Then this exponent will equal the magnetic flux defined in (3.4), plus an additional contribution $\phi(\mathbf{x}, \mathbf{x})$ which is zero due to the antisymmetry of the magnetic phase. Now if we expand $e^{i\delta\mathbf{F}^1_j(\mathbf{x}, \dots, \mathbf{y}_j)}$ in Taylor series, combine this with (3.27), after some algebra we eventually get (3.26). Then we can identify the n th derivative at ω_0 of the kernel's diagonal as the coefficient multiplying the n th power of $\delta\omega$. The theorem is proved. \square .

The natural question is why is formula (3.12) better than the one from (2.34)? The answer is that $W_{k,L}^m(\mathbf{x}; \beta, \omega_0)$ *does not grow with L* , and we will see in the next section that it even converges when L tends to infinity. Let us show here its uniform boundedness in L .

Looking at its definition given in (3.11), and using the estimates from (3.9) together with the diamagnetic inequality, we see that we need to estimate

$$\begin{aligned} |W_{k,L}^m(\mathbf{x}; \beta, \omega)| &\leq C_3^k \sum_{j=1}^k (-1)^j \sum_{(i_1, \dots, i_j) \in \{1,2\}^j} \chi_j^k(i_1, \dots, i_j) \int_{D_j(\beta)} d\tau \int_{\Lambda^j} d\mathbf{y} \\ &\frac{|\text{Fl}_j(\mathbf{x}, \mathbf{y}_1, \dots, \mathbf{y}_j)|^m}{m!} G_\infty(\mathbf{x}, \mathbf{y}_1; 16(\beta - \tau_1)) G_\infty(\mathbf{y}_1, \mathbf{y}_2; 16(\tau_1 - \tau_2)) \\ &\dots G_\infty(\mathbf{y}_{j-1}, \mathbf{y}_j; 16(\tau_{j-1} - \tau_j)) G_\infty(\mathbf{y}_j, \mathbf{x}; 16\tau_j) \end{aligned} \quad (3.29)$$

Let $\alpha = 16m$, and identify $\mathbf{x} = \mathbf{y}_0$. In view of (3.5) and the explicit form (2.29) of the heat kernel, for $1 \leq l \leq j-1, 1 \leq l' \leq l; \mathbf{y}_0, \mathbf{y}_1, \dots, \mathbf{y}_j \in \Lambda_L^{j+1}$ and $\{\tau_1, \tau_2, \dots, \tau_j\} \in D_j(\beta)$ we need the straightforward estimate

$$\begin{aligned} |\mathbf{y}_{l'-1} - \mathbf{y}_{l'}| |\mathbf{y}_l - \mathbf{y}_{l+1}| \exp\left(-\frac{|\mathbf{x} - \mathbf{y}_1|^2}{2\alpha(\beta - \tau_1)}\right) \dots \exp\left(-\frac{|\mathbf{y}_k - \mathbf{x}|^2}{2\alpha\tau_k}\right) \\ \leq 2\alpha\beta \exp\left(-\frac{|\mathbf{x} - \mathbf{y}_1|^2}{4\alpha(\beta - \tau_1)}\right) \dots \exp\left(-\frac{|\mathbf{y}_k - \mathbf{x}|^2}{4\alpha\tau_k}\right). \end{aligned} \quad (3.30)$$

Thus (3.5) and (3.30) imply

$$\begin{aligned} &|\text{Fl}_j(\mathbf{x}, \mathbf{y}_1, \dots, \mathbf{y}_j)|^m G_\infty(\mathbf{x}, \mathbf{y}_1; 16(\beta - \tau_1)) \dots G_\infty(\mathbf{y}_j, \mathbf{x}; 16\tau_j) \\ &\leq \left(\sum_{l=1}^{j-1} l\alpha\beta\right)^m 2^{3(j+1)/2} G_\infty(\mathbf{x}, \mathbf{y}_1; 32(\beta - \tau_1)) \dots G_\infty(\mathbf{y}_j, \mathbf{x}; 32\tau_j) \\ &\leq (8j^2 m\beta)^m 2^{3(j+1)/2} G_\infty(\mathbf{x}, \mathbf{y}_1; 32(\beta - \tau_1)) \dots G_\infty(\mathbf{y}_j, \mathbf{x}; 32\tau_j). \end{aligned} \quad (3.31)$$

Integrating over the spatial coordinates, using the semigroup property, and then integrating over τ variables, one eventually obtains the uniform upper bound in L :

$$|W_{k,L}^m(\mathbf{x}; \beta, \omega)| \leq c(m, k) \frac{(1 + \beta)^{7(m+k)+3}}{\beta^{3/2}} (1 + \omega)^{3(m+k)+2} \quad (3.32)$$

where $c(m, k) = (m + k)! c^{m+k} \frac{m^m}{m!} \sum_{j=1}^k \frac{j^{2m}}{j!}$ and c is again a numerical factor.

Remark 3.4. *The Theorem 3.1 gives us what we need for the purpose of this article. One can show that our analysis can be applied in order to get the off-diagonal terms of the integral kernel i.e. $\frac{\partial^n G_L}{\partial \omega^n}(\mathbf{x}, \mathbf{x}'; \beta, \omega)$, $(\mathbf{x}, \mathbf{x}') \in \Lambda_L \times \Lambda_L$ and $\beta > 0, \omega > 0$. In that case we get*

$$\begin{aligned}
& \frac{1}{n!} \frac{\partial^n G_L}{\partial \omega^n}(\mathbf{x}, \mathbf{x}'; \beta, \omega) = (-1)^n \frac{\phi^n(\mathbf{x}, \mathbf{x}')}{n!} G_L(\mathbf{x}, \mathbf{x}'; \beta, \omega) \\
& + \sum_{k=1}^n \sum_{j=1}^k (-1)^j \int_{D_j(\beta)} d\tau \sum_{(i_1, \dots, i_j) \in \{1, 2\}^j} \chi_j^j(i_1, \dots, i_j) \int_{\Lambda^j} d\mathbf{y} \\
& \cdot \frac{\left(i \left(\sum_{l=1}^{j-1} \text{fl}(\mathbf{x}, \mathbf{y}_l, y_{l+1}) + \text{fl}(\mathbf{x}, \mathbf{y}_j, \mathbf{x}') + \phi(\mathbf{x}, \mathbf{x}') \right) \right)^{N-m}}{(N-m)!} \\
& \cdot G_L(\mathbf{x}, \mathbf{y}_1, \beta - \tau_1, \omega) R_{i_1, L}(\mathbf{y}_1, \mathbf{y}_2, \tau_1 - \tau_2, \omega) \dots R_{i_k, L}(\mathbf{y}_k, \mathbf{x}', \tau_k, \omega). \quad (3.33)
\end{aligned}$$

4 Large volume behavior

For further applications in Section 4 we need to have a similar formula to the Theorem 3.1 when $L = \infty$. But the results of Section 2 cannot be directly applied to this situation. In spite of this fact, we will show in this section that Theorem 3.1 remains true even if we take $L = \infty$, and the quantities at finite volume converge pointwise to the ones defined on the whole space.

Recall first that the explicit form of the integral kernel of $e^{-\beta H_\infty(\omega)}$; $\beta > 0, \omega \geq 0$ is given by

$$\begin{aligned}
G_\infty(\mathbf{x}, \mathbf{x}'; \beta, \omega) &= \frac{1}{(2\pi\beta)^{3/2}} \frac{\omega\beta/2}{\sinh(\omega\beta/2)} e^{i\omega\phi(\mathbf{x}, \mathbf{x}')} \\
&\cdot \exp \left\{ -\frac{1}{2\beta} \left(\frac{\omega\beta/2}{\tanh(\omega\beta/2)} [(x_1 - x'_1)^2 + (x_2 - x'_2)^2] + (x_3 - x'_3)^2 \right) \right\}
\end{aligned} \quad (4.1)$$

where the phase ϕ is defined in (3.1).

We start with a technical result. For any $\mathbf{x} \in \Lambda_L$, we denote with $d(\mathbf{x}) := \text{dist}(\mathbf{x}, \partial\Lambda_L)$. Let $M := \{(\mathbf{x}, \mathbf{x}') \in \Lambda_L \times \Lambda_L : d(\mathbf{x}) \leq 1 \text{ or } d(\mathbf{x}') \leq 1\}$, and denote with χ_M the characteristic function of M .

Theorem 4.1. *Let $\beta > 0$ and $\omega \geq 0$. Then for any $(\mathbf{x}, \mathbf{x}') \in \Lambda_L \times \Lambda_L$ we have:*

$$\begin{aligned}
& |G_L(\mathbf{x}, \mathbf{x}'; \beta, \omega) - G_\infty(\mathbf{x}, \mathbf{x}'; \beta, \omega)| \leq 2\chi_M(\mathbf{x}, \mathbf{x}') G_\infty(\mathbf{x}, \mathbf{x}'; \beta) \\
& + \mathcal{C}_4(1 - \chi_M)(\mathbf{x}, \mathbf{x}') G_\infty(\mathbf{x}, \mathbf{x}'; 16\beta) e^{-\left(\frac{d^2(\mathbf{x})}{64\beta} + \frac{d^2(\mathbf{x}')}{64\beta}\right)},
\end{aligned} \quad (4.2)$$

and

$$\begin{aligned}
& |(-i\nabla_{\mathbf{x}} - \omega\mathbf{a}(\mathbf{x}))[G_L(\mathbf{x}, \mathbf{x}'; \beta, \omega) - G_\infty(\mathbf{x}, \mathbf{x}'; \beta, \omega)]| \\
& \leq \frac{\mathcal{C}_5}{\sqrt{\beta}} \chi_M(\mathbf{x}, \mathbf{x}') G_\infty(\mathbf{x}, \mathbf{x}'; 8\beta) + \mathcal{C}_6(1 - \chi_M) G_\infty(\mathbf{x}, \mathbf{x}'; 16\beta) e^{-\left(\frac{d^2(\mathbf{x})}{64\beta} + \frac{d^2(\mathbf{x}')}{64\beta}\right)}.
\end{aligned} \quad (4.3)$$

where $\mathcal{C}_4 = \mathcal{C}_4(\beta, \omega) = c(1 + \beta)^7(1 + \omega)^4$, $\mathcal{C}_5 = \mathcal{C}_5(\beta, \omega) = c(1 + \beta)^5(1 + \omega)^3$, $\mathcal{C}_6 = \mathcal{C}_6(\beta, \omega) = c(1 + \beta)^8(1 + \omega)^5$ and $c > 1$ is a numerical constant.

To prove the theorem, we need the following lemma.

Lemma 4.2. *Let $\beta > 0$, $\omega \geq 0$ and $\alpha \in \{0, 1\}$. Then for every $(\mathbf{x}, \mathbf{x}') \in \Lambda_L \times \Lambda_L$ we have:*

$$\begin{aligned} & \partial_{x_i}^\alpha G_L(\mathbf{x}, \mathbf{x}'; \beta, \omega) - \partial_{x_i}^\alpha G_\infty(\mathbf{x}, \mathbf{x}'; \beta, \omega) \\ & \quad \frac{1}{2} \int_0^\beta d\tau \int_{\partial\Lambda_L} d\sigma(\mathbf{y}) \partial_{x_i}^\alpha G_\infty(\mathbf{x}, \mathbf{y}; \tau, \omega) [\mathbf{n}_\mathbf{y} \cdot \nabla_\mathbf{y} G_L(\mathbf{y}, \mathbf{x}'; \beta - \tau, \omega)], \end{aligned} \quad (4.4)$$

where $d\sigma(\mathbf{y})$ is the measure on $\partial\Lambda_L$ and $\mathbf{n}_\mathbf{y}$ is the outer normal to $\partial\Lambda_L$ at \mathbf{y} .

Proof. Let $\beta > 0$, $\omega \geq 0$. Recall that both Green functions $G_\infty(\mathbf{x}, \mathbf{x}'; \tau, \omega)$ and $G_L(\mathbf{x}, \mathbf{x}'; \tau, \omega)$ satisfy in $\Lambda_L \times \Lambda_L$ in distributional sense the equation

$$(i) \quad \partial_\tau G(\mathbf{x}, \mathbf{x}'; \tau) = -\frac{1}{2} [-i\nabla_\mathbf{x} - \omega \mathbf{a}(\mathbf{x})]^2 G(\mathbf{x}, \mathbf{x}'; \tau); \quad \tau > 0 \quad (4.5)$$

$$(ii) \quad G(\mathbf{x}, \mathbf{x}'; \tau = 0_+) = \delta(\mathbf{x} - \mathbf{x}'). \quad (4.6)$$

For $0 < \tau < \beta$ and on $\Lambda_L \times \Lambda_L$, define the following quantity:

$$Q(\mathbf{x}, \mathbf{x}'; \beta, \tau) := \int_{\Lambda_L} d\mathbf{y} \overline{G_L(\mathbf{y}, \mathbf{x}; \beta - \tau, \omega)} G_\infty(\mathbf{y}, \mathbf{x}'; \tau, \omega). \quad (4.7)$$

Then by (4.5) it is easy to see that

$$\begin{aligned} & \partial_\tau Q(\mathbf{x}, \mathbf{x}'; \beta, \tau) \\ & = \frac{1}{2} \int_{\Lambda_L} d\mathbf{y} \left\{ \overline{(-i\nabla_\mathbf{y} - \omega \mathbf{a}(\mathbf{y}))^2 G_L(\mathbf{y}, \mathbf{x}; \beta - \tau, \omega)} G_\infty(\mathbf{y}, \mathbf{x}'; \tau, \omega) \right. \\ & \quad \left. - \overline{G_L(\mathbf{y}, \mathbf{x}; \beta - \tau, \omega)} (-i\nabla_\mathbf{y} - \omega \mathbf{a}(\mathbf{y}))^2 G_\infty(\mathbf{y}, \mathbf{x}'; \tau, \omega) \right\}. \end{aligned} \quad (4.8)$$

Since $G_L(\mathbf{x}, \mathbf{x}'; \tau, \omega) = 0$ if $\mathbf{x} \in \partial\Lambda_L$ or $\mathbf{x}' \in \partial\Lambda_L$, integration by parts gives:

$$\partial_\tau Q(\mathbf{x}, \mathbf{x}'; \beta, \tau) = -\frac{1}{2} \int_{\partial\Lambda_L} d\sigma(\mathbf{y}) \mathbf{n}_\mathbf{y} \cdot \nabla_\mathbf{y} \overline{G_L(\mathbf{y}, \mathbf{x}; \beta - \tau, \omega)} G_\infty(\mathbf{y}, \mathbf{x}'; \tau, \omega). \quad (4.9)$$

Now by integrating with respect to τ from 0_+ to β_- , and using (4.6), we obtain:

$$\begin{aligned} & G_L(\mathbf{x}, \mathbf{x}'; \beta, \omega) - G_\infty(\mathbf{x}, \mathbf{x}'; \beta, \omega) \\ & = \frac{1}{2} \int_0^\beta d\tau \int_{\partial\Lambda_L} d\sigma(\mathbf{y}) \mathbf{n}_\mathbf{y} \cdot \nabla_\mathbf{y} \overline{G_L(\mathbf{y}, \mathbf{x}; \beta - \tau, \omega)} G_\infty(\mathbf{y}, \mathbf{x}'; \tau, \omega). \end{aligned} \quad (4.10)$$

Now using the self-adjointness property of the semigroup we obtain $G(\mathbf{x}, \mathbf{y}; \tau) = \overline{G(\mathbf{y}, \mathbf{x}; \tau)}$, thus we can rewrite (4.10) as:

$$\begin{aligned} & G_L(\mathbf{x}, \mathbf{x}'; \beta, \omega) - G_\infty(\mathbf{x}, \mathbf{x}'; \beta, \omega) \\ & = \frac{1}{2} \int_0^\beta d\tau \int_{\partial\Lambda_L} d\sigma(\mathbf{y}) G_\infty(\mathbf{x}, \mathbf{y}; \tau, \omega) [\mathbf{n}_\mathbf{y} \cdot \nabla_\mathbf{y} G_L(\mathbf{y}, \mathbf{x}'; \beta - \tau, \omega)]. \end{aligned} \quad (4.11)$$

The lemma now follows from (4.11). \square

Proof of Theorem 4.1. Let $\beta > 0$, $\omega \geq 0$ and suppose first that $(\mathbf{x}, \mathbf{x}') \in M$. Then (4.2) follows from the diamagnetic inequality (2.30). Let us show (4.3) in the same case. We know from Lemma 2.5, (2.31) that

$$|(-i\nabla_{\mathbf{x}} - \omega \mathbf{a}(\mathbf{x}))G_L(\mathbf{x}, \mathbf{x}'; \beta, \omega)| \leq \frac{\mathcal{C}_1}{\sqrt{\beta}} G_{\infty}(\mathbf{x}, \mathbf{x}'; 8\beta). \quad (4.12)$$

On the other hand, using the observation that $-i\nabla_{\mathbf{x}} - \omega \mathbf{a}(\mathbf{x})$ is transformed into $-i\nabla_{\mathbf{x}} - \omega \mathbf{a}(\mathbf{x} - \mathbf{x}')$ after commutation with $e^{i\phi(\mathbf{x}, \mathbf{x}')}$, then by direct computation from (4.1) we get that for all $\eta > 0$,

$$|(-i\nabla_{\mathbf{x}} - \omega \mathbf{a}(\mathbf{x}))G_{\infty}(\mathbf{x}, \mathbf{x}'; \eta\beta, \omega)| \leq \frac{\mathcal{C}'_1}{\sqrt{\eta\beta}} G_{\infty}(\mathbf{x}, \mathbf{x}'; 2\eta\beta), \quad (4.13)$$

where $\mathcal{C}'_1 = \mathcal{C}'_1(\beta, \omega) = 2(1 + \eta)(1 + \omega)(1 + \beta)$. Then (4.12) and (4.13) for $\eta = 1$ imply (4.3).

Now suppose that $(\mathbf{x}, \mathbf{x}') \in M^c$. This means that neither points are near the boundary. For $\mathbf{y} \in \partial\Lambda$ then by (2.31) we have:

$$|\nabla_{\mathbf{y}} G_L(\mathbf{y}, \mathbf{x}; \beta - \tau, \omega)| \leq \frac{\mathcal{C}_1}{\sqrt{\beta - \tau}} G_{\infty}(\mathbf{y}, \mathbf{x}, 8(\beta - \tau)). \quad (4.14)$$

By applying the estimates (4.14), (2.30) and the Lemma 4.2, we get

$$\begin{aligned} & |G_L(\mathbf{x}, \mathbf{x}'; \beta, \omega) - G_{\infty}(\mathbf{x}, \mathbf{x}'; \beta, \omega)| \\ & \leq 2^{7/2} \mathcal{C}_1 \int_0^{\beta} d\tau \int_{\partial\Lambda_L} d\sigma(\mathbf{y}) \frac{G_{\infty}(\mathbf{y}, \mathbf{x}; 8(\beta - \tau))}{\sqrt{\beta - \tau}} G_{\infty}(\mathbf{y}, \mathbf{x}'; 8\tau). \end{aligned} \quad (4.15)$$

But if $\mathbf{y} \in \partial\Lambda_L$, $|\mathbf{x} - \mathbf{y}| \geq d(\mathbf{x})$, then a straightforward estimate shows that for $0 < t < \beta$, we have $G_{\infty}(\mathbf{y}, \mathbf{x}; 8t) \leq e^{-\frac{d^2(\mathbf{x})}{32\beta}} G_{\infty}(\mathbf{y}, \mathbf{x}; 16t)$. Thus we get

$$\begin{aligned} & \text{r.h.s. of (4.15)} \\ & \leq 2^5 \mathcal{C}_1 \int_0^{\beta} d\tau \frac{e^{-\left(\frac{d^2(\mathbf{x})}{32(\beta - \tau)} + \frac{d^2(\mathbf{x}')}{32\tau}\right)}}{\sqrt{\beta - \tau}} \int_{\partial\Lambda_L} d\sigma(\mathbf{y}) G_{\infty}(\mathbf{y}, \mathbf{x}; 16(\beta - \tau)) G_{\infty}(\mathbf{y}, \mathbf{x}'; 16\tau). \end{aligned} \quad (4.16)$$

For any $t, t' > 0$, let us look at the integral

$$\int_{\partial\Lambda_L} d\sigma(\mathbf{y}) G_{\infty}(\mathbf{y}, \mathbf{x}; t) G_{\infty}(\mathbf{y}, \mathbf{x}'; t'). \quad (4.17)$$

Using the convexity of Λ_L , replacing the integrals on the sides of $\partial\Lambda_L$ by integrals on \mathbb{R}^2 (thus getting an upper bound), and using the semigroup property in two dimensions, we can show that there exists a numerical constant $C > 0$ such that

$$\int_{\partial\Lambda_L} d\sigma(\mathbf{y}) G_{\infty}(\mathbf{y}, \mathbf{x}; t) G_{\infty}(\mathbf{y}, \mathbf{x}'; t') \leq C \frac{\sqrt{t+t'}}{\sqrt{t}\sqrt{t'}} G_{\infty}(\mathbf{x}, \mathbf{x}'; t+t'). \quad (4.18)$$

To be more precise, let us look at the integral on the hyperplane defined by $H := \mathbb{R}^2 + (L/2, 0, 0)$:

$$\int_H d\sigma(\mathbf{y}) G_{\infty}(\mathbf{y}, \mathbf{x}; t) G_{\infty}(\mathbf{y}, \mathbf{x}'; t'), \quad (4.19)$$

where \mathbf{x} and \mathbf{x}' are on the same side of \mathbb{R}^3 with respect to H . Decompose $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$ and $\mathbf{x}' = \mathbf{x}'_1 + \mathbf{x}'_2$ where \mathbf{x}_1 and \mathbf{x}'_1 are the parallel components with H , while \mathbf{x}_2 and \mathbf{x}'_2 are the orthogonal components on H . Note that here $|\mathbf{x}_2|^2 + |\mathbf{x}'_2|^2 \geq |\mathbf{x}_2 - \mathbf{x}'_2|^2$. Since $|\mathbf{x} - \mathbf{y}|^2 = |\mathbf{x}_1 - \mathbf{y}|^2 + |\mathbf{x}_2|^2$, we can explicitly integrate with respect to \mathbf{y} and eventually get (4.18).

Then we can write:

$$\int_{\partial\Lambda_L} d\sigma(\mathbf{y}) G_\infty(\mathbf{y}, \mathbf{x}; 16(\beta - \tau)) G_\infty(\mathbf{y}, \mathbf{x}'; 16\tau) \leq \mathcal{C}'_1 \frac{\sqrt{\beta}}{\sqrt{(\beta - \tau)\tau}} G_\infty(\mathbf{x}, \mathbf{x}'; 16\beta).$$

Therefore since \mathbf{x}, \mathbf{x}' satisfy $d(\mathbf{x}), d(\mathbf{x}') \geq 1$ we get

$$\begin{aligned} |G_L(\mathbf{x}, \mathbf{x}'; \beta, \omega) - G_\infty(\mathbf{x}, \mathbf{x}'; \beta, \omega)| &\leq 2^4 \mathcal{C}'_1 G_\infty(\mathbf{x}, \mathbf{x}'; 16\tau) \\ &\cdot e^{-\left(\frac{d^2(\mathbf{x})}{64\beta} + \frac{d^2(\mathbf{x}')}{64\beta}\right)} \int_0^\beta d\tau \frac{e^{-\left(\frac{1}{64(\beta-\tau)} + \frac{1}{64\tau}\right)}}{\sqrt{\beta-\tau}} \frac{1}{\sqrt{(\beta-\tau)\tau}}. \end{aligned} \quad (4.20)$$

Due the exponential decay, there are no singularities in this integral, and a straightforward estimate gives (4.2).

We now use the same method as above to prove (4.3) in the case when $(\mathbf{x}, \mathbf{x}') \in M^c$. We know from Lemma 4.2 that

$$\begin{aligned} (-i\nabla_{\mathbf{x}} - \omega \mathbf{a}(x)) (G_L(\mathbf{x}, \mathbf{x}'; \beta, \omega) - G_\infty(\mathbf{x}, \mathbf{x}'; \beta, \omega)) &= \frac{1}{2} \int_0^\beta d\tau \int_{\partial\Lambda_L} d\sigma(\mathbf{y}) \\ &\cdot (-i\nabla_{\mathbf{x}} - \omega \mathbf{a}(x)) G_\infty(\mathbf{x}, \mathbf{y}; \tau, \omega) [\mathbf{n}_{\mathbf{y}} \cdot \nabla_{\mathbf{y}} G_L(\mathbf{y}, \mathbf{x}'; \beta - \tau, \omega)]. \end{aligned} \quad (4.21)$$

Then by (4.12) and (4.13)

$$\begin{aligned} |(-i\nabla_{\mathbf{x}} - \omega \mathbf{a}(x)) (G_L(\mathbf{x}, \mathbf{x}'; \beta, \omega) - G_\infty(\mathbf{x}, \mathbf{x}'; \beta, \omega))| &\leq \\ 4\mathcal{C}'_1 \int_0^\beta d\tau \int_{\partial\Lambda_L} d\sigma(\mathbf{y}) \frac{G_\infty(\mathbf{y}, \mathbf{x}; 8(\beta - \tau))}{\sqrt{\beta - \tau}} \frac{G_\infty(\mathbf{y}, \mathbf{x}'; 8\tau)}{\sqrt{\tau}}. \end{aligned} \quad (4.22)$$

Then by using the same arguments leading to (4.20) we get

$$\begin{aligned} |(-i\nabla_{\mathbf{x}} - \omega \mathbf{a}(x)) (G_L(\mathbf{x}, \mathbf{x}'; \beta, \omega) - G_\infty(\mathbf{x}, \mathbf{x}'; \beta, \omega))| &\leq \\ \leq 16\mathcal{C}'_1{}^2 G_\infty(\mathbf{x}, \mathbf{x}'; 16\beta) e^{-\left(\frac{d^2(\mathbf{x})}{64\beta} + \frac{d^2(\mathbf{x}')}{64\beta}\right)} \int_0^\beta d\tau \frac{e^{-\left(\frac{1}{64(\beta-\tau)} + \frac{1}{64\tau}\right)}}{\tau(\beta - \tau)}, \end{aligned} \quad (4.23)$$

from which (4.3) follows. Theorem 4.1 is proved. \square

We now want to prove that the equality (3.12) stated in Theorem 3.1 remains true even if L tends to infinity. It is well known (see e.g. [AC]) that for $\beta > 0, \omega \geq 0$ and $(\mathbf{x}, \mathbf{x}') \in \mathbb{R}^3 \times \mathbb{R}^3$,

$$G_\infty(\mathbf{x}, \mathbf{x}'; \beta, \omega) = \lim_{L \rightarrow \infty} G_L(\mathbf{x}, \mathbf{x}'; \beta, \omega). \quad (4.24)$$

Our main goal now is to show that this pointwise convergence holds true for all the derivatives $\frac{\partial^n G_L}{\partial \omega^n}$, $n \geq 1$.

We need to introduce some notation. Let $\beta > 0$ and $\omega \geq 0$. For $(\mathbf{x}, \mathbf{x}') \in \mathbb{R}^3 \times \mathbb{R}^3$ define

$$\begin{aligned} R_{1,\infty}(\mathbf{x}, \mathbf{x}'; \beta, \omega) &:= \mathbf{a}(\mathbf{x} - \mathbf{x}') \cdot (i\nabla_{\mathbf{x}} + \omega \mathbf{a}(\mathbf{x})) G_\infty(\mathbf{x}, \mathbf{x}'; \beta, \omega). \\ R_{2,\infty}(\mathbf{x}, \mathbf{x}'; \beta, \omega) &:= \frac{1}{2} \mathbf{a}^2(\mathbf{x} - \mathbf{x}') G_\infty(\mathbf{x}, \mathbf{x}'; \beta, \omega). \end{aligned} \quad (4.25)$$

Let us note that we again have the same type of estimates as in (3.7), (3.8) and (3.9), i.e. gaussian localization in the difference of the spatial arguments. The linear growth of the magnetic potential disappears when one commutes $-i\nabla_{\mathbf{x}}$ with the magnetic phase, as we have already seen in (4.13).

Now define for $\mathbf{x} \in \mathbb{R}^3$, $k \geq 1, m \geq 0$:

$$\begin{aligned} W_{k,\infty}^m(\mathbf{x}; \beta, \omega) &:= \sum_{j=1}^k (-1)^j \sum_{(i_1, \dots, i_j) \in \{1,2\}^j} \chi_j^k(i_1, \dots, i_j) \int_{D_j(\beta)} d\tau \int_{\mathbb{R}^{3j}} d\mathbf{y} \\ &\frac{(i(\text{Fl}_j(\mathbf{x}, \mathbf{y}_1, \dots, \mathbf{y}_j)))^m}{m!} G_{\infty}(\mathbf{x}, \mathbf{y}_1; \beta - \tau_1, \omega) R_{i_1, \infty}(\mathbf{y}_1, \mathbf{y}_2; \tau_1 - \tau_2, \omega) \\ &\dots R_{i_{j-1}, \infty}(\mathbf{y}_{j-1}, \mathbf{y}_j; \tau_{j-1} - \tau_j, \omega) R_{i_j, \infty}(\mathbf{y}_j, \mathbf{x}; \tau_j, \omega). \end{aligned} \quad (4.26)$$

Since every integrand is bounded by a free heat kernel, and because the flux Fl_j can be bound by *differences* of its arguments (see (3.5)), then the above multiple integrals are absolutely convergent. Also note the important thing that multiplication by $|\mathbf{y} - \mathbf{y}'|^m$ of the free heat kernel only improves the singularity in the time variable due to the estimate

$$|\mathbf{y} - \mathbf{y}'|^m \mathbf{e}^{-|\mathbf{y} - \mathbf{y}'|^2/t} \leq \text{const} \cdot t^{m/2} \mathbf{e}^{-|\mathbf{y} - \mathbf{y}'|^2/(2t)}. \quad (4.27)$$

The last important remark about $W_{k,\infty}^m(\mathbf{x}; \beta, \omega)$ is that *it does not depend on \mathbf{x}* . This can be seen by factorizing all the magnetic phases which enter in the various factors of the integrand, and see that they add up to give another Fl_j , which only depends on differences of variables. The remaining factors are also just functions of differences of variables. Therefore by changing \mathbf{x} we get the same value for $W_{k,\infty}^m$ after a change of variables (a translation) in all integrals.

Then we have

Theorem 4.3. *Let $\beta > 0$ and $\omega \geq 0$. Fix $\mathbf{x} \in \mathbb{R}^3$ and $n \geq 1$. Then we have:*

$$\frac{1}{n!} \frac{\partial^n G_{\infty}}{\partial \omega^n}(\mathbf{x}, \mathbf{x}; \beta, \omega) = \lim_{L \rightarrow \infty} \frac{1}{n!} \frac{\partial^n G_L}{\partial \omega^n}(\mathbf{x}, \mathbf{x}; \beta, \omega) = \sum_{k=1}^n W_{k,\infty}^{n-k}(\mathbf{x}; \beta, \omega). \quad (4.28)$$

Proof. Fix $\beta > 0$ and $\omega \geq 0$. Let $n \geq 1$ and $(\mathbf{x}, \mathbf{x}') \in \mathbb{R}^3$. Choose L large enough such that $\mathbf{x} \in \Lambda_L$. Then from (4.2) and (4.3) we have:

$$|G_L(\mathbf{x}, \mathbf{x}'; \beta, \omega) - G_{\infty}(\mathbf{x}, \mathbf{x}'; \beta, \omega)| \leq C_4 \beta^{-3/2} e^{-(\frac{d^2(\mathbf{x})}{64\beta} + \frac{d^2(\mathbf{x}')}{64\beta})} G_{\infty}(\mathbf{x}, \mathbf{x}'; 16\beta), \quad (4.29)$$

$$|R_{1,L}(\mathbf{x}, \mathbf{x}'; \beta, \omega) - R_{1,\infty}(\mathbf{x}, \mathbf{x}'; \beta, \omega)| \leq C_6 \beta^{-1} e^{-(\frac{d^2(\mathbf{x})}{64\beta} + \frac{d^2(\mathbf{x}')}{64\beta})} G_{\infty}(\mathbf{x}, \mathbf{x}'; 16\beta),$$

$$|R_{2,L}(\mathbf{x}, \mathbf{x}'; \beta, \omega) - R_{2,\infty}(\mathbf{x}, \mathbf{x}'; \beta, \omega)| \leq C_4 \beta^{-1/2} e^{-(\frac{d^2(\mathbf{x})}{64\beta} + \frac{d^2(\mathbf{x}')}{64\beta})} G_{\infty}(\mathbf{x}, \mathbf{x}'; 16\beta).$$

Then for all $(\mathbf{x}, \mathbf{x}') \in \mathbb{R}^3 \times \mathbb{R}^3$, estimates (4.29) show respectively that

$$\lim_{L \rightarrow \infty} G_L(\mathbf{x}, \mathbf{x}', \beta, \omega) = G_{\infty}(\mathbf{x}, \mathbf{x}', \beta, \omega),$$

$$\lim_{L \rightarrow \infty} R_{i,L}(\mathbf{x}, \mathbf{x}', \beta, \omega) = R_{i,\infty}(\mathbf{x}, \mathbf{x}', \beta, \omega),$$

for $i = 1, 2$. Then

$$\lim_{L \rightarrow \infty} G_L(\mathbf{x}, \mathbf{y}_1; \beta - \tau_1, \omega) R_{i_1,L}(\mathbf{y}_1, \mathbf{y}_2; \tau_1 - \tau_2, \omega) \dots R_{i_j,L}(\mathbf{y}_j, \mathbf{x}; \tau_j, \omega) =$$

$$G_\infty(\mathbf{x}, \mathbf{y}_1; \beta - \tau_1, \omega) R_{i_1, \infty}(\mathbf{y}_1, \mathbf{y}_2; \tau_1 - \tau_2, \omega) \dots R_{i_j, L}(\mathbf{y}_j, \mathbf{x}; \tau_j, \omega).$$

Furthermore by (3.9) and (2.30) we have

$$|G_L(\mathbf{x}, \mathbf{y}_1; \beta - \tau_1, \omega) R_{i_1, L}(\mathbf{y}_1, \mathbf{y}_2; \tau_1 - \tau_2, \omega) \dots R_{i_j, L}(\mathbf{y}_j, \mathbf{x}; \tau_j, \omega)| \leq$$

$$4^3 \mathcal{C}_3^j G_\infty(\mathbf{x}, \mathbf{y}_1; 16(\beta - \tau_1)) G_\infty(\mathbf{y}_1, \mathbf{y}_2; 16(\tau_1 - \tau_2)) \dots G_\infty(\mathbf{y}_j, \mathbf{x}; 16\tau_j).$$

this last quantity is L -independent and \mathbb{R}^3 -integrable by the semigroup property since

$$\int_{\mathbb{R}^{3j}} d\mathbf{y} G_\infty(\mathbf{x}, \mathbf{y}_1; 16(\beta - \tau_1)) \dots G_\infty(\mathbf{y}_j, \mathbf{x}; 16\tau_j) = G_\infty(\mathbf{x}, \mathbf{y}_1; 16\beta).$$

Note that the flux Fl_j does not influence anything, since it can be bound by powers of differences between spatial variables, which will meet the gaussian decay of the free heat kernels. Thus they will only affect the time integrals (by making them even less singular).

Then by applying Lebesgue's dominated convergence theorem we get from (3.11) and (3.12),

$$\lim_{L \rightarrow \infty} \frac{1}{n!} \frac{\partial^n G_L}{\partial \omega^n}(\mathbf{x}, \mathbf{x}, \beta, \omega) = \sum_{k=1}^n W_{k, \infty}^{n-k}(\mathbf{x}; \beta, \omega). \quad (4.30)$$

Now the remaining thing is to show that this also equals $\frac{1}{n!} \frac{\partial^n G_\infty}{\partial \omega^n}(\mathbf{x}, \mathbf{x}, \beta, \omega)$. Fix $\omega_0 \geq 0$ and choose $\omega \in \mathbb{R}$ such that $|\delta\omega| = |\omega - \omega_0| \leq 1$. From the usual Taylor formula we can write

$$\begin{aligned} G_L(\mathbf{x}, \mathbf{x}, \beta, \omega) &= \sum_{n=0}^N (\delta\omega)^n \frac{1}{n!} \frac{\partial^n G_L}{\partial \omega^n}(\mathbf{x}, \mathbf{x}, \beta, \omega_0) \\ &\quad + (\delta\omega)^{N+1} \frac{1}{(N+1)!} \frac{\partial^{N+1} G_L}{\partial \omega^{N+1}}(\mathbf{x}, \mathbf{x}, \beta, \omega_1), \end{aligned} \quad (4.31)$$

where ω_1 is between ω_0 and ω . Then by taking L to infinity, we easily get the estimate (note that $\sum_{k=1}^{N+1} W_{k, L}^{N+1-k}(\mathbf{x}; \beta, \omega_1)$ is bounded by a constant independent of L , see the estimate from (3.32)) :

$$\begin{aligned} \left| G_\infty(\mathbf{x}, \mathbf{x}, \beta, \omega) - G_\infty(\mathbf{x}, \mathbf{x}, \beta, \omega_0) - \sum_{n=1}^N (\delta\omega)^n \sum_{k=1}^n W_{k, \infty}^{n-k}(\mathbf{x}; \beta, \omega_0) \right| \\ \leq C(N, \beta) |\delta\omega|^{N+1}. \end{aligned} \quad (4.32)$$

Since $G_\infty(\mathbf{x}, \mathbf{x}; \beta, \omega)$ is smooth in ω (see (1.7)), it follows that the coefficient of $(\delta\omega)^n$ must equal $\frac{1}{n!} \frac{\partial^n G_\infty}{\partial \omega^n}(\mathbf{x}, \mathbf{x}, \beta, \omega_0)$, and we are done. \square

5 Thermodynamic limit for magnetic susceptibilities

As a consequence of the analysis of the previous section we are now able to prove the main technical result of this paper:

Theorem 5.1. Let $n \geq 1$, $\beta > 0$ and $\omega \geq 0$.

$$\int_{\Lambda_L} d\mathbf{x} \left| \frac{\partial^n G_\infty}{\partial \omega^n}(\mathbf{x}, \mathbf{x}; \beta, \omega) - \frac{\partial^n G_L}{\partial \omega^n}(\mathbf{x}, \mathbf{x}; \beta, \omega) \right| \leq L^2 \mathcal{C}(n, \beta, \omega). \quad (5.1)$$

where $\mathcal{C}(n, \beta, \omega) := c(n) \frac{(1+\beta)^{7n+8}}{\sqrt{\beta}} (1+\omega)^{3n+5}$ where $c(n)$ only depends on n .

Proof. Let $1 \leq j \leq k$, $1 \leq k \leq n$, $1 \leq m \leq n - k$, and denote the integrand in (4.26) with:

$$\begin{aligned} F_{j,\infty}^m &= F_{j,\infty}^m(\mathbf{x}, \mathbf{y}_1, \dots, \mathbf{y}_j, \tau_1, \dots, \tau_j, \beta, \omega) \\ &= \frac{(i\text{Fl}_j(\mathbf{x}, \mathbf{y}_1, \dots, \mathbf{y}_j))^m}{m!} G_\infty(\mathbf{x}, \mathbf{y}_1, \beta - \tau_1, \omega) \dots R_{i_j, \infty}(\mathbf{y}_j, \mathbf{x}, \tau_j, \omega). \end{aligned} \quad (5.2)$$

Denote also by:

$$\begin{aligned} F_{j,L}^m &= F_{j,L}^m(\mathbf{x}, \mathbf{y}_1, \dots, \mathbf{y}_j, \tau_1, \dots, \tau_j, \beta, \omega) \\ &= \frac{(i\text{Fl}_j(\mathbf{x}, \mathbf{y}_1, \dots, \mathbf{y}_j))^m}{m!} G_L(\mathbf{x}, \mathbf{y}_1, \beta - \tau_1, \omega) \dots R_{i_j, L}(\mathbf{y}_j, \mathbf{x}, \tau_j, \omega). \end{aligned} \quad (5.3)$$

Let $n \geq 1$, $\beta \geq 0$, $\omega \geq 0$, and fix $\mathbf{x} \in \mathbb{R}^3$. Then by applying the Theorem 4.3, we can split the integrals from W 's in "inner" and "outer" regions:

$$\frac{\partial^n G_\infty}{\partial \omega^n}(\mathbf{x}, \mathbf{x}, \beta, \omega) = f_L^n(\mathbf{x}, \beta, \omega) + g_L^n(\mathbf{x}, \beta, \omega) \quad (5.4)$$

where

$$\begin{aligned} f_L^n(\mathbf{x}, \beta, \omega) &:= n! \sum_{k=1}^n \sum_{j=1}^k (-1)^j \sum_{(i_1, \dots, i_j) \in \{1,2\}^j} \chi_j^k(i_1, \dots, i_j) \int_{D_j(\beta)} d\tau \int_{\Lambda_L^j} d\mathbf{y} \\ &\quad F_{j,\infty}^{n-k}(\mathbf{x}, \mathbf{y}_1, \dots, \mathbf{y}_j, \tau_1, \dots, \tau_j, \beta, \omega), \end{aligned} \quad (5.5)$$

$$\begin{aligned} g_L^n(\mathbf{x}, \beta, \omega) &:= n! \sum_{k=1}^n \sum_{j=1}^k (-1)^j \sum_{(i_1, \dots, i_j) \in \{1,2\}^j} \chi_j^k(i_1, \dots, i_j) \sum_{l=1}^j \int_{D_j(\beta)} d\tau \\ &\quad \int_{\mathbb{R}^3} d\mathbf{y}_1 \dots \int_{\mathbb{R}^3 \setminus \Lambda_L} d\mathbf{y}_l \int_{\mathbb{R}^3} d\mathbf{y}_{l+1} \dots \int_{\mathbb{R}^3} d\mathbf{y}_j F_{j,\infty}^{n-k}(\mathbf{x}, \mathbf{y}_1, \dots, \mathbf{y}_j, \tau_1, \dots, \tau_j, \beta, \omega). \end{aligned} \quad (5.6)$$

Let us now show that

$$\int_{\Lambda} d\mathbf{x} \left| f_L^n(\mathbf{x}, \beta, \omega) - \frac{\partial^n G_L}{\partial \omega^n}(\mathbf{x}, \mathbf{x}; \beta, \omega) \right| \leq L^2 f(n, \beta, \omega), \quad (5.7)$$

where $f(n, \beta, \omega) := c(n) \frac{(1+\beta)^{7n+3}}{\sqrt{\beta}} (1+\omega)^{3n+2}$, $c(n)$ depending only on n .

From now, for the sake of simplicity we often omit the explicit dependence of all variables. In view of (3.11), (3.12) and (4.26), (4.28), we need to estimate

$$\begin{aligned} F_{j,\infty}^m - F_{j,L}^m &= \frac{(i\text{Fl}_j)^m}{m!} \{ (G_\infty - G_L) R_{i_1, \infty} \dots R_{i_j, \infty} + \\ &\quad \sum_{l=1}^j G_L R_{i_1, L} \dots R_{i_{l-1}, L} (R_{i_l, \infty} - R_{i_l, L}) R_{i_{l+1}, \infty} \dots R_{i_j, \infty} \}. \end{aligned} \quad (5.8)$$

Denote by $\chi(\mathbf{x})$ the characteristic of $\{\mathbf{x} \in \Lambda, d(\mathbf{x}) \leq 1\}$. Thanks to the Theorem 4.1, we have

$$\begin{aligned} & |(R_{1,\infty} - R_{1,L})(\mathbf{x}, \mathbf{x}'; \beta, \omega)| \\ & \leq |\mathbf{a}(\mathbf{x} - \mathbf{x}')(-i\nabla_{\mathbf{x}} - \omega\mathbf{a})(G_{\infty} - G_L)(\mathbf{x}, \mathbf{x}'; \beta, \omega)| \\ & \leq \mathcal{C}_7 G_{\infty}(\mathbf{x}, \mathbf{x}'; 32\beta) \left(\chi(\mathbf{x}) + \chi(\mathbf{x}') + e^{-\left(\frac{d^2(\mathbf{x})}{64\beta} + \frac{d^2(\mathbf{x}')}{64\beta}\right)} \right), \end{aligned} \quad (5.9)$$

where $\mathcal{C}_7 = \mathcal{C}_7(\beta, \omega) = c(1 + \beta)^9(1 + \omega)^5$ for some numerical constant $c > 1$. But again by the Theorem 4.1 we may use the bound

$$\begin{aligned} & |(G_{\infty} - G_L)(\mathbf{x}, \mathbf{x}'; \beta, \omega)|, |(R_{2,\infty} - R_{2,L})(\mathbf{x}, \mathbf{x}'; \beta, \omega)| \leq \\ & \mathcal{C}_7 G_{\infty}(\mathbf{x}, \mathbf{x}'; 32\beta) \left(\chi(\mathbf{x}) + \chi(\mathbf{x}') + e^{-\left(\frac{d^2(\mathbf{x})}{64\beta} + \frac{d^2(\mathbf{x}')}{64\beta}\right)} \right). \end{aligned} \quad (5.10)$$

On the other hand by (3.9), (4.13) and (4.25), the kernel of $R_{i,\infty}$, $i = 1, 2$ and of $R_{i,L}$, $i = 1, 2$ satisfy the inequality

$$\max \{|(R_{i,\infty}(\mathbf{x}, \mathbf{x}'; \beta, \omega)|, |R_{i,L}(\mathbf{x}, \mathbf{x}'; \beta, \omega)|\} \leq \mathcal{C}'_3 G_{\infty}(\mathbf{x}, \mathbf{x}'; 16\beta), \quad (5.11)$$

where $\mathcal{C}'_3 = \mathcal{C}'_3(\beta, \omega) = c \cdot \mathcal{C}_1$, \mathcal{C}_1 is defined in (2.37) and $c > 1$ is a numerical constant which is chosen large enough such that we have $G_{\infty}(\mathbf{x}, \mathbf{x}'; \beta) \leq \mathcal{C}'_3 G_{\infty}(\mathbf{x}, \mathbf{x}'; 16\beta)$.

Replace \mathbf{x} with some $\mathbf{y}_0 \in \mathbb{R}^3$. Then (2.30), (5.11) together with (5.9) and (5.10) give

$$\begin{aligned} |F_{j,\infty}^m - F_{j,L}^m| & \leq \mathcal{C}_7 \mathcal{C}'_3^{j-1} \frac{|\mathbf{F}|_j|^m}{m!} G_{\infty}(\mathbf{y}_0, \mathbf{y}_1; 32(\beta - \tau_1)) \dots \\ & G_{\infty}(\mathbf{y}_j, \mathbf{y}_0; 32\tau_j) \sum_{l=0}^j (2\chi(\mathbf{x}_l) + e^{-\frac{d^2(\mathbf{x}_l)}{64\beta}}). \end{aligned} \quad (5.12)$$

Thus from this inequality and (3.5), we need to estimate the quantity

$$Q := \left(\sum_{l=1}^{j-1} \sum_{l'=1}^l |\mathbf{y}_{l'-1} - \mathbf{y}_{l'}| |\mathbf{y}_l - \mathbf{y}_{l+1}| \right)^m G_{\infty}(\mathbf{y}_0, \mathbf{y}_1; 32(\beta - \tau_1)) \dots G_{\infty}(\mathbf{y}_j, \mathbf{y}_0; 32\tau_j).$$

By using (4.27) we have

$$Q \leq (8\beta j^2)^m m^m 2^{3(j+1)/2} G_{\infty}(\mathbf{y}_0, \mathbf{y}_1; 64(\beta - \tau_1)) \dots G_{\infty}(\mathbf{y}_j, \mathbf{y}_0; 64\tau_j).$$

and then for $j \leq n$

$$\begin{aligned} |F_{j,\infty}^m - F_{j,L}^m| & \leq 2^{3(n+1)/2} \mathcal{C}_7 \mathcal{C}'_3^{m-1} \frac{(8\beta j^2)^m m^m}{m!} G_{\infty}(\mathbf{y}_0, \mathbf{y}_1; 64(\beta - \tau_1)) \dots \\ & G_{\infty}(\mathbf{y}_j, \mathbf{y}_0; 64\tau_j) \sum_{l=0}^j (2\chi(\mathbf{x}_l) + e^{-\frac{d^2(\mathbf{x}_l)}{64\beta}}). \end{aligned} \quad (5.13)$$

By extending the integration with respect to $\mathbf{y}_0, \dots, \mathbf{y}_{l-1}, \mathbf{y}_{l+1}, \dots, \mathbf{y}_j$ on the whole \mathbb{R}^3 space, and using the semigroup property (2.40) and the fact that $G_{\infty}(\mathbf{x}, \mathbf{x}; t) =$

$\frac{1}{(2\pi t)^{3/2}}$ we get

$$\left| \int_{\Lambda_L^{j+1}} d\mathbf{y} (F_{j,\infty}^{n-k} - F_{j,L}^{n-k}) \right| \leq c^n \mathcal{C}_7 \mathcal{C}'_3^{n-1} (j+1) \frac{(\beta j^2)^{(n-k)} (n-k)^{n-k}}{\beta^{3/2} (n-k)!} \quad (5.14)$$

$$\cdot \int_{\Lambda_L} d\mathbf{x} (2\chi(\mathbf{x}) + e^{-\frac{d^2(\mathbf{x})}{64\beta}})$$

for some positive constant c . Moreover, simple estimates show that

$$\int_{\Lambda_L} d\mathbf{x} (2\chi(\mathbf{x}) + e^{-\frac{d^2(\mathbf{x})}{64\beta}}) \leq cL^2(1 + \sqrt{\beta}),$$

where c is also some positive numerical constant. From (5.5) and the Theorem 3.1,

$$\begin{aligned} & \int_{\Lambda_L} d\mathbf{y}_0 \left\{ \frac{\partial^n G_L}{\partial \omega^n}(\mathbf{y}_0, \mathbf{y}_0; \beta, \omega) - f_L^n(\mathbf{y}_0, \beta, \omega) \right\} \quad (5.15) \\ &= n! \sum_{k=1}^n \sum_{j=1}^k (-1)^j \sum_{(i_1, \dots, i_j) \in \{1,2\}^j} \chi_j^k(i_1, \dots, i_j) \\ & \cdot \int_{D_j(\beta)} d\tau \int_{\Lambda^{j+1}} d\mathbf{y} (F_{j,\infty}^{n-k} - F_{j,L}^{n-k})(\mathbf{y}_0, \mathbf{y}_1, \dots, \mathbf{y}_j, \tau_1, \dots, \tau_j, \beta, \omega). \end{aligned}$$

Then (5.14) together with (5.15) lead to:

$$\left| \int_{\Lambda_L} d\mathbf{y}_0 \left[\frac{\partial^n G_L}{\partial \omega^n}(\mathbf{y}_0, \mathbf{x}; \beta, \omega) - f_L^n(\mathbf{y}_0, \beta, \omega) \right] \right| \leq L^2 c(n) \frac{(1+\beta)^{7n+3}}{\sqrt{\beta}} (1+\omega)^{3n+2}$$

where $c(n) = (n+1)! c^n \sum_{k=1}^n \frac{(n-k)^{n-k}}{(n-k)!} \sum_{j=1}^k \frac{j^{2(n-k)}}{j!}$ and c is again a numerical factor. This last estimate clearly implies (5.7).

Let us now prove that for all $\beta > 0$ and $\omega \geq 0$, $g_L^n(\mathbf{y}_0, \beta, \omega)$ given in (5.6) satisfies:

$$\left| \int_{\Lambda_L} d\mathbf{y}_0 g_L^n(\mathbf{y}_0, \beta, \omega) \right| \leq L^2 g(n, \beta, \omega), \quad (5.16)$$

where $g(n, \beta, \omega) := c(n)(1+\beta)^{7n+2}(1+\omega)^{3n+2}$ and $c(n)$ is a positive constant depending only on n . The same arguments as above leading to the estimate (5.13) imply

$$\begin{aligned} & |F_{j,\infty}^m(\mathbf{y}_0, \dots, \mathbf{y}_j, \tau_1, \dots, \tau_j, \beta, \omega)| \quad (5.17) \\ & \leq \mathcal{C}_7 \mathcal{C}'_3^{j-1} \frac{|\text{Fl}_j|^m}{m!} G_\infty(\mathbf{y}_0, \mathbf{y}_1; 32(\beta - \tau_1)) \dots G_\infty(\mathbf{y}_j, \mathbf{y}_0; 32\tau_j) \\ & \leq \mathcal{C}_7 \mathcal{C}'_3^{j-1} \frac{(8\beta j^2)^m m^m 2^{3(j+1)/2}}{m!} G_\infty(\mathbf{y}_0, \mathbf{y}_1; 64(\beta - \tau_1)) \dots G_\infty(\mathbf{y}_j, \mathbf{y}_0; 64\tau_j). \end{aligned}$$

On the other hand, by the semigroup property (put $\Lambda_L^c := \mathbb{R}^3 \setminus \Lambda_L$),

$$\begin{aligned} & \int_{\mathbb{R}^3} d\mathbf{y}_1 \dots \int_{\Lambda_L^c} d\mathbf{y}_l \int_{\mathbb{R}^3} d\mathbf{y}_{l+1} \dots \int_{\mathbb{R}^3} d\mathbf{y}_j G_\infty(\mathbf{y}_0, \mathbf{y}_1; 64(\beta - \tau_1)) \dots G_\infty(\mathbf{y}_j, \mathbf{y}_0; 64\tau_j) \\ &= \int_{\Lambda_L^c} d\mathbf{y}_l G_\infty(\mathbf{y}_0, \mathbf{y}_l; 64(\beta - \tau_l)) G_\infty(\mathbf{y}_l, \mathbf{y}_0; 64\tau_l). \quad (5.18) \end{aligned}$$

Then (5.6), (5.17) and (5.18) imply

$$\left| \int_{\Lambda_L} d\mathbf{y}_0 g_L^n(\mathbf{y}_0, \beta, \omega) \right| \leq n! \sum_{k=1}^n \sum_{j=1}^k \mathcal{C}_7 \mathcal{C}'_3^{j-1} 2^{(5j+3)/2} \frac{(8\beta j^2)^{(n-k)} (n-k)^{n-k}}{(n-k)!} \\ \cdot \int_{D_j(\beta)} d\tau \sum_{l=1}^j \int_{\Lambda_L} d\mathbf{y}_0 \int_{\Lambda_L^c} d\mathbf{y}_l G_\infty(\mathbf{y}_0, \mathbf{y}_l; 64(\beta - \tau_l)) G_\infty(\mathbf{y}_l, \mathbf{y}_0; 64\tau_l). \quad (5.19)$$

By using the explicit form of the heat kernel given in (2.29), a straightforward computation shows that

$$\int_{\Lambda_L} d\mathbf{y}_0 \int_{\Lambda_L^c} d\mathbf{y}_l G_\infty(\mathbf{y}_0, \mathbf{y}_l; 64(\beta - \tau_l)) G_\infty(\mathbf{y}_l, \mathbf{y}_0; 64\tau_l) \leq c \frac{L^2}{\beta}$$

for some positive constant c . Hence we get

$$\left| \int_{\Lambda} d\mathbf{y}_0 g_L^n(\mathbf{y}_0, \beta, \omega) \right| \leq L^2 c(n) (1 + \beta)^{7n+2} (1 + \omega)^{3n+2}$$

where $c(n) = n! c^n \sum_{k=1}^n \sum_{j=1}^k \frac{j^{2(n-k)} (n-k)^{n-k}}{(n-k)!(j-1)!}$ and c is a positive numerical factor. This shows (5.16). Then (5.16) and (5.7) imply the theorem. \square

5.1 The proof of Theorem 1.2

We are now ready to prove the thermodynamic limit of generalized susceptibilities in the grandcanonical ensemble, when the chemical potential is negative (fugacity z less than one).

Let $L \geq 1$, $\beta > 0$, $\omega \geq 0$ and $|z| < 1$. We know from (1.8) and (2.27) that:

$$P_L(\beta, \omega, z, \epsilon) - P_\infty(\beta, \omega, z, \epsilon) \\ = \frac{\epsilon}{\beta |\Lambda_L|} \sum_{k \geq 1} \frac{(-\epsilon z)^k}{k} \int_{\Lambda_L} d\mathbf{x} \{ G_L(\mathbf{x}, \mathbf{x}; k\beta, \omega, z) - G_\infty(\mathbf{x}, \mathbf{x}; k\beta, \omega, z) \}. \quad (5.20)$$

Then by applying the Theorem 5.1 we get

$$\frac{\partial^n (P_L - P_\infty)}{\partial \omega^n} = \frac{\epsilon}{\beta |\Lambda_L|} \sum_{k \geq 1} \frac{(-\epsilon z)^k}{k} \int_{\Lambda_L} d\mathbf{x} \left(\frac{\partial^n G_L(\mathbf{x}, \mathbf{x})}{\partial \omega^n} - \frac{\partial^n G_\infty(\mathbf{x}, \mathbf{x})}{\partial \omega^n} \right).$$

In particular, this also shows that the series from (1.10) must converge. Moreover by using again the bound (5.1) in the last formula, we have

$$|\chi_L^{(n)} - \chi_\infty^{(n)}| \leq c(n) (1 + \omega)^{3n+5} \frac{1}{\beta L} \sum_{k \geq 1} \frac{|z|^k}{k} \frac{(1 + k\beta)^{7n+8}}{\sqrt{k\beta}}.$$

Since the series in the r.h.s of this last inequality is finite and L independent, this proves (1.11). \square

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